

Quantized meson fields in and out of equilibrium. I : Kinetics of meson condensate and quasi-particle excitations

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Abstract

We formulate a kinetic theory of self-interacting meson fields with an aim to describe the freezeout stage of the space-time evolution of matter in ultrarelativistic nuclear collisions. Kinetic equations are obtained from the Heisenberg equation of motion for a single component real scalar quantum field taking the mean field approximation for the non-linear interaction. The mesonic mean field obeys the classical non-linear Klein-Gordon equation with a modification due to the coupling to mesonic quasi-particle excitations which are expressed in terms of the Wigner functions of the quantum fluctuations of the meson field, namely the statistical average of the bilinear forms of the meson creation and annihilation operators. In the long wavelength limit, the equations of motion of the diagonal components of the Wigner functions take a form of Vlasov equation with a particle source and sink which arises due to the non-vanishing off-diagonal components of the Wigner function expressing coherent pair-creation and pair-annihilation process in the presence of non-uniform condensate. We show that in the static homogeneous system, these kinetic equations reduce to the well-known gap equation in the Hartree approximation, and hence they may be considered as a generalization of the Hartree approximation method to non-equilibrium systems. As an application of these kinetic equations, we compute the dispersion relations of the collective mesonic excitations in the system near equilibrium.

Key words: meson condensate; Vlasov equation

1 Introduction

Theoretical study of the space-time evolution of matter in high energy nuclear collisions has a long history since the early pioneering works of Fermi

and Landau employing thermodynamics and fluid mechanics [1]. More recent works [2] have been motivated by the prospect of studying a new form of matter experimentally by means of very energetic collisions of heavy nuclei as those currently underway at Brookhaven with Relativistic Heavy-Ion Collider (RHIC) and planned at CERN with Large Hadron Collider (LHC). The data taken from the RHIC experiments has shown existence of a strong anisotropic (elliptic) collective flow of hadrons in non-central collisions [3], indicating early local equilibration of dense matter produced by the collision [4]. This supports the hydrodynamic picture of matter evolution and gives us a hope to learn information of the equation of state of dense matter from systematic analyses of the data.

Much attention has been paid to the early stage of matter evolution where a dense plasma of unconfined quarks and gluons has been expected to be formed through some complex non-equilibrium processes [5]. One anticipates also a breakdown of the hydrodynamic behavior of the system when it is diluted sufficiently and the collision time exceeds the characteristic time scale of expansion. This stage of the matter evolution is usually referred to as the freezeout stage. The aim of this work is to present a kinetic theory which is designed to describe the freezeout stage of the expanding hadron gas.

The freezeout of the expanding hadronic matter would proceed in several steps; chemical freezeout of the relative abundance of hadron species may occur before the kinetic freezeout of the momentum distribution of hadrons. Observed relative abundance of hadrons fit very nicely to a simple picture that the chemical freezeout occurs at a certain temperature and baryon chemical potential [6, 3]. Some of the other observables, such as two particle momentum correlations, the pion analogue of the Hanbury Brown and Twiss two photon intensity interferometry, may be very sensitive to the dynamics of the final kinetic freezeout as indicated by the “HBT puzzle” found in the recent data analysis [7].

If the initial state of the dense matter formed in the collision is a plasma of deconfined quarks and gluons, freezeout of the color degrees of freedom should proceed these processes when the quark-gluon plasma hadronizes. Chiral symmetry [8], an approximate global symmetry of QCD which becomes exact in the limit of vanishing quark masses and is considered to be broken spontaneously in the QCD vacuum, may also play an important role in the freezeout dynamics.

As a dense hadronic matter formed by ultrarelativistic nuclear collision is diluted by the expansion, one expects that the system undergoes a phase transition associated with the spontaneous breakdown of the chiral symmetry which is restored temporally after the collision by the formation of a quark-gluon plasma. As the quark-gluon plasma hadronizes and the system turns into the

confining phase, the system would gradually develop a vacuum chiral condensate and remaining excitations would expand in the influence of the growing chiral condensate. This physical picture has been elaborated in terms of a classical equation of motion for the chiral condensate; effects of excitations were described in terms of statistical fluctuations in the classical fields [9, 10]. The fluctuating condensate described by classical pion field has been termed Disoriented Chiral Condensate (DCC), emphasizing the symmetry aspect of the problem [10]. What was missing in these classical treatments of the meson fields is the existence of particle excitations in addition to the condensate. Inclusion of particle excitations requires the quantization of the fields. The effect of the quantum fluctuations of meson fields has been studied by Tsue, Vautherin and one of the present authors (TM) [11] by the functional Schrödinger picture formalism [12, 13].

The physical picture we have described is very similar to what happens when the dilute gas of magnetically trapped atoms of alkali metals cools down by evaporation [14]. Some of the atoms condense into the lowest single particle level in the trapping external potential forming a Bose-Einstein condensate. The dynamics of such a system may be described by the coupled equations of motion for the condensate, or the Gross-Pitaevskii equation [15], and the kinetic equation, or the Boltzmann-Vlasov equations, for the phase space distribution of the excitations in the presence of the condensate [16, 17].

In this paper we will show that a similar set of equations can be derived for a system of interacting mesons described by the relativistic quantum field theory by the mean field approximation. This approximation corresponds to a neglect of all correlations in the system [18]. We will make no attempt in this work to justify this approximation and leave it as an open problem for further study.

In this context we note that field theoretical derivation of a Boltzmann-type kinetic equation has been given by many others [19]. Most of these works focuses on the derivation of the collision terms. The present work is distinguished from these works in the emphasis of the role of the mean field in the evolution of the system coupled with the quasi-particle excitations in the same spirit as in [13, 11]. We omit the effect of the quasi-particle collisions in this work. This procedure may be reasonable, at least as a first step, for describing dynamical aspect of the freeze-out process: even in the absence of collisions, interactions between the quasi-particles and the evolving condensate would affect the final particle distribution. But the present approach is not adequate, however, for the early thermalization problem, where the collision terms play essential role [5].¹ We adopt the standard Heisenberg picture instead of the functional

¹ We note that the mean fields may also play important role in the thermalization problem, for example via the non-Abelian analogue of the plasma instability [20], or by the anomalous enhancement of collision rates in a turbulent plasma evolution [21].

Schrödinger picture used in [13, 11] since we found it more straightforward to see the connection to familiar semi-classical kinetic equation. In this first of a series of papers we shall use one-component real scalar field model interacting via a ϕ^4 self-interaction term in order to concentrate on the presentation of the basic features of the theory. Analysis with multi-component scalar model with $O(N)$ symmetry will be deferred for the forthcoming paper [22].

In the next section we formulate a quantum kinetic theory for quantum scalar meson field, starting from the Heisenberg equations for quantum field of one-component real scalar. The mean field approximation replaces the products of quantum fields by the products of classical fields and the statistical average of bilinear forms of the quantum fluctuations. The latter is expressed in terms of the Wigner functions which is reduced to the single particle distribution function in the classical Boltzmann equation. Our theory contains another forms of the Wigner functions which have no classical counter parts and arises due to the coherent pair creation and annihilation processes in non-uniform systems. Only in uniform systems, these *off-diagonal* components of the fluctuation can be eliminated by suitable redefinition of the particle mass. Appearance of the off-diagonal Wigner functions is reminiscent of the anomalous propagators in the microscopic theory of superconductivity [23, 24, 25]. Similar structure also appears in the theory of Bose-Einstein Condensate [17]. Some details of the mean field calculation is given in Appendix.

In section 3, we apply our method to uniform systems and show that each mode characterized by the particle momentum obeys non-linear forced oscillatory motion. In section 4 we show that in equilibrium these kinetic equations are reduced to a gap equation [26] which determine the equilibrium amplitude of the condensate and the mass parameter as a function of the temperature. In this paper, we ignore the effect of the divergent vacuum polarizations which requires a subtle renormalization procedure in the mean field approximation [27]. The solution exhibits characteristic features of the first order phase transition.

In section 5, we study slowly varying non-uniform systems. Taking long wavelength approximation, the equations of motion of the diagonal components of the Wigner functions are reduced to a Vlasov equation in a form generalized by Landau for quasi-particles excitations in quantum Fermi liquid [28, 29] with the quasi-particle energy given in the mean field approximation. Non-vanishing off-diagonal components of the Wigner functions generate extra terms in the Vlasov equation which may be interpreted as particle source and sink terms.

In section 6, we compute the dispersion relations of excitations of the system near equilibrium. Solving the coupled kinetic equations by linearizing the equations with respect to small oscillatory deviations from equilibrium solution we obtain dispersion relation of the excitation modes in the system near equilibrium. We find the continuum of quasi-particle excitations in the

entire space-like energy-momentum region in addition to the continuum in time-like region due to the (thermally induced) pair creation. We found that in the low temperature phase the meson pole shifts due to the coupling to the quasi-particle continua. The effective meson mass vanishes at the edge of the spinodal instability line of the first order transition.² We also found that the coupling of the off-diagonal components of the Wigner function plays an important role to prevent appearance of undamped tachyonic sound mode which propagates with a velocity greater than that of light.

A short summary of the paper is given in section 7 with remarks on the remaining problems.

2 Kinetic equations for the meson condensate and quasi-particle excitations

In this section we derive quantum kinetic equations which describes the time evolution of the meson condensate coupled with mesonic quasi-particle excitations. We use the natural unit $\hbar = c = 1$ throughout this paper.

2.1 Quantized real scalar field in the Heisenberg representation

We first take a simple model of a self-interacting real scalar field in the Heisenberg picture. The Hamiltonian is given by

$$H = \int d\mathbf{r} \left[\frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \hat{\phi})^2 + \mathcal{V}[\hat{\phi}] \right] \quad (1)$$

where

$$\mathcal{V}[\hat{\phi}] = \frac{1}{2} m^2 \hat{\phi}^2 + \frac{\lambda}{4!} \hat{\phi}^4. \quad (2)$$

Here the scalar field $\hat{\phi}$ and its canonical conjugate momentum field $\hat{\pi}$ are quantized by the equal-time commutation relations:

$$[\hat{\phi}(\mathbf{r}, t), \hat{\pi}(\mathbf{r}', t)] = i\delta(\mathbf{r} - \mathbf{r}') \quad (3)$$

$$[\hat{\phi}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)] = [\hat{\pi}(\mathbf{r}, t), \hat{\pi}(\mathbf{r}', t)] = 0 \quad (4)$$

² Softening of the σ meson mode has been studied first by Hatsuda and Kunihiro as a precursory phenomenon of the chiral phase transition using the Nambu-Jona-Lasinio model with quark fields.[30]

The Heisenberg equation of motion of the quantum field $\hat{\phi}(\mathbf{r}, t)$ is given by

$$\frac{\partial \hat{\phi}}{\partial t} = -i[\hat{\phi}, H] = \hat{\pi}(\mathbf{r}, t) \quad (5)$$

while the equation of motion of the canonical conjugate field $\hat{\pi}(\mathbf{r}, t)$ becomes

$$\frac{\partial \hat{\pi}}{\partial t} = -i[\hat{\pi}, H] = (\nabla^2 - m^2)\hat{\phi}(\mathbf{r}, t) - \frac{1}{3!}\lambda\hat{\phi}^3(\mathbf{r}, t) \quad (6)$$

Elimination of the field momentum $\hat{\pi}$ from these equations yields a modified Klein-Gordon equation for the quantum scalar field $\hat{\phi}$:

$$\square\hat{\phi}(\mathbf{r}, t) + m^2\hat{\phi}(\mathbf{r}, t) = -\frac{1}{3!}\lambda\hat{\phi}^3(\mathbf{r}, t) \quad (7)$$

where $\square = \partial^2/\partial t^2 - \nabla^2$.

2.2 Density matrix and Gaussian Ansatz

We are interested in the time evolution of the system described by the density operator

$$\hat{\rho} = \sum_s |\Psi_s\rangle p_s \langle \Psi_s| \quad (8)$$

where $\{|\Psi_s\rangle\}$ are a set of normalized wave functions and p_s is the probability distribution for a mixed state described by this density matrix so that it satisfies

$$\sum_s p_s = 1 \quad (9)$$

This density matrix can be also expressed in terms of some complete set $|\alpha\rangle$ of the wave functions of our Hilbert-Fock space as

$$\hat{\rho} = \sum_{\alpha, \beta} |\alpha\rangle \rho_{\alpha\beta} \langle \beta| \quad (10)$$

where

$$\rho_{\alpha\beta} = \sum_s \langle \alpha | \Psi_s \rangle p_s \langle \Psi_s | \beta \rangle \quad (11)$$

All physical information of the system to be described are contained in a specific form of the density matrix. In thermodynamic equilibrium, the density matrix is given by

$$\hat{\rho}_{\text{eq}} = Z^{-1} e^{-\beta H} \quad (12)$$

with

$$Z = \text{tr} [e^{-\beta H}] = e^{\beta F(T)} \quad (13)$$

where $F(T)$ gives the Helmholtz free energy of the system at temperature $T = 1/\beta$.

In the Heisenberg picture the density matrix is time-independent since the wave functions are time-independent; all time dependence arises from the time dependence of an operator:

$$\langle \hat{\mathcal{O}}(t) \rangle = \text{tr} [\hat{\mathcal{O}}(t) \hat{\rho}] \quad (14)$$

For example, we define the classical condensate fields by the statistical average of the quantum fields,

$$\phi_c(\mathbf{r}, t) = \langle \hat{\phi}(\mathbf{r}, t) \rangle, \quad (15)$$

$$\pi_c(\mathbf{r}, t) = \langle \hat{\pi}(\mathbf{r}, t) \rangle \quad (16)$$

In the following we take a Gaussian Ansatz for the density matrix:

$$\langle \tilde{\phi}^n(\mathbf{r}, t) \rangle = 0 \quad \text{for odd integer } n \quad (17)$$

and

$$\langle \tilde{\phi}^n(\mathbf{r}, t) \rangle = \frac{n!}{m! 2^m} \langle \tilde{\phi}^2(\mathbf{r}, t) \rangle^m \quad \text{for even integer } n = 2m \quad (18)$$

where $\tilde{\phi}(\mathbf{r}, t)$ and $\tilde{\pi}(\mathbf{r}, t)$ are shifted field operators defined by

$$\tilde{\phi}(\mathbf{r}, t) = \hat{\phi}(\mathbf{r}, t) - \phi_c(\mathbf{r}, t) \quad (19)$$

$$\tilde{\pi}(\mathbf{r}, t) = \hat{\pi}(\mathbf{r}, t) - \pi_c(\mathbf{r}, t) \quad (20)$$

The shifted field operators obey the same equal-time commutation relations as the original fields:

$$[\tilde{\phi}(\mathbf{r}, t), \tilde{\pi}(\mathbf{r}', t)] = i\delta(\mathbf{r} - \mathbf{r}') \quad (21)$$

$$[\tilde{\phi}(\mathbf{r}, t), \tilde{\phi}(\mathbf{r}', t)] = [\tilde{\pi}(\mathbf{r}, t), \tilde{\pi}(\mathbf{r}', t)] = 0 \quad (22)$$

We will show that this choice of the density matrix is a non-equilibrium generalization of the Hartree approximation in equilibrium. It has been alternatively introduced in the functional Schrödinger representation [13, 11]

2.3 The Wigner functions

We introduce the one-particle Wigner function by

$$F(\mathbf{p}, \mathbf{k}, t) = \langle a_{\mathbf{p}+\mathbf{k}/2}^\dagger(t) a_{\mathbf{p}-\mathbf{k}/2}(t) \rangle \quad (23)$$

Here the particle creation and annihilation operators may be defined in terms of the Fourier transforms of the shifted fields

$$\tilde{\phi}_{\mathbf{p}}(t) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \tilde{\phi}(\mathbf{r}, t), \quad \tilde{\pi}_{\mathbf{p}}(t) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \tilde{\pi}(\mathbf{r}, t) \quad (24)$$

as

$$a_{\mathbf{p}}(t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} [\omega_{\mathbf{p}} \tilde{\phi}_{\mathbf{p}}(t) + i\tilde{\pi}_{-\mathbf{p}}(t)] \quad (25)$$

$$a_{\mathbf{p}}^\dagger(t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} [\omega_{\mathbf{p}} \tilde{\phi}_{\mathbf{p}}(t) - i\tilde{\pi}_{-\mathbf{p}}(t)] \quad (26)$$

with

$$\omega_{\mathbf{p}} = \sqrt{p^2 + \mu^2} \quad (27)$$

These relations are rewritten as:

$$\tilde{\phi}(\mathbf{r}, t) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} [a_{\mathbf{p}}(t) + a_{-\mathbf{p}}^\dagger(t)] \quad (28)$$

$$\tilde{\pi}(\mathbf{r}, t) = i \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} [a_{\mathbf{p}}^\dagger(t) - a_{-\mathbf{p}}(t)] \quad (29)$$

The quantization rules (21, 22) are transcribed to

$$[a_{\mathbf{p}}(t), a_{\mathbf{p}'}^\dagger(t)] = \delta_{\mathbf{p}, \mathbf{p}'} \quad (30)$$

$$[a_{\mathbf{p}}(t), a_{\mathbf{p}'}(t)] = [a_{\mathbf{p}}^\dagger(t), a_{\mathbf{p}'}^\dagger(t)] = 0. \quad (31)$$

with which we may interpret $a_{\mathbf{p}}$ ($a_{\mathbf{p}}^\dagger$) as annihilation (creation) operator of “particle excitation” with momentum \mathbf{p} .

We also introduce the following other forms of the Wigner functions:

$$G(\mathbf{p}, \mathbf{k}, t) = \langle a_{-\mathbf{p}-\mathbf{k}/2}(t) a_{\mathbf{p}-\mathbf{k}/2}(t) \rangle, \quad (32)$$

$$\bar{G}(\mathbf{p}, \mathbf{k}, t) = \langle a_{\mathbf{p}+\mathbf{k}/2}^\dagger(t) a_{-\mathbf{p}+\mathbf{k}/2}^\dagger(t) \rangle, \quad (33)$$

$$\bar{F}(\mathbf{p}, \mathbf{k}, t) = \langle a_{-\mathbf{p}-\mathbf{k}/2}(t) a_{-\mathbf{p}+\mathbf{k}/2}^\dagger(t) \rangle, \quad (34)$$

These Wigner functions are not independent but are related to each other. The complex conjugate of the Wigner functions are given by

$$F^*(\mathbf{p}, \mathbf{k}, t) = F(\mathbf{p}, -\mathbf{k}, t), \quad (35)$$

$$\bar{F}^*(\mathbf{p}, \mathbf{k}, t) = \bar{F}(\mathbf{p}, -\mathbf{k}, t) \quad (36)$$

$$\bar{G}(\mathbf{p}, \mathbf{k}, t) = G^*(\mathbf{p}, -\mathbf{k}, t) \quad (37)$$

where the asterisk (*) stands for the complex conjugate. The commutation relations imply also that $G(\mathbf{p}, \mathbf{k}, t)$ and $\bar{G}(\mathbf{p}, \mathbf{k}, t)$ are even functions of \mathbf{p}

$$G(\mathbf{p}, \mathbf{k}, t) = G(-\mathbf{p}, \mathbf{k}, t) \quad (38)$$

$$\bar{G}(\mathbf{p}, \mathbf{k}, t) = \bar{G}(-\mathbf{p}, \mathbf{k}, t) \quad (39)$$

and

$$\bar{F}(\mathbf{p}, \mathbf{k}, t) = F(-\mathbf{p}, \mathbf{k}, t) + \delta_{\mathbf{k},0} \quad (40)$$

The four Wigner functions may be grouped together to form a matrix form of the Wigner function

$$\mathbf{W}(\mathbf{p}, \mathbf{k}, t) = \begin{pmatrix} F(\mathbf{p}, \mathbf{k}, t) & \bar{G}(\mathbf{p}, \mathbf{k}, t) \\ G(\mathbf{p}, \mathbf{k}, t) & \bar{F}(\mathbf{p}, \mathbf{k}, t) \end{pmatrix} \quad (41)$$

The appearance of the “off-diagonal” components of the Wigner functions is reminiscent of the anomalous propagators in the BCS theory of superconductivity which arises due to the presence of fermion pair condensate [23, 24, 25]. Note that our definition of the matrix components of the Wigner function is slightly different from those for the propagators.

We write the Fourier transforms of the Wigner functions as

$$f(\mathbf{p}, \mathbf{r}, t) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{p}, \mathbf{k}, t), \quad \bar{f}(\mathbf{p}, \mathbf{r}, t) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \bar{F}(\mathbf{p}, \mathbf{k}, t), \quad (42)$$

$$g(\mathbf{p}, \mathbf{r}, t) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} G(\mathbf{p}, \mathbf{k}, t), \quad \bar{g}(\mathbf{p}, \mathbf{r}, t) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \bar{G}(\mathbf{p}, \mathbf{k}, t) \quad (43)$$

The (35) and (36) imply that $f(\mathbf{p}, \mathbf{r}, t)$ and $\bar{f}(\mathbf{p}, \mathbf{r}, t)$ are real functions and are related to each other by

$$\bar{f}(\mathbf{p}, \mathbf{r}, t) = f(-\mathbf{p}, \mathbf{r}, t) + 1, \quad (44)$$

while (37) implies that $g(\mathbf{p}, \mathbf{r}, t)$ and $\bar{g}(\mathbf{p}, \mathbf{r}, t)$ are complex conjugate to each other:

$$g^*(\mathbf{p}, \mathbf{r}, t) = \bar{g}(\mathbf{p}, \mathbf{r}, t) \quad (45)$$

Here we have chosen the particle “mass” μ to be different from the mass parameter m in the original Hamiltonian. Physical particle mass for interacting fields is generally different from the mass parameter in the Hamiltonian or Lagrangian due to the effect of interaction, e.g. renormalization with or without the spontaneous symmetry breaking. It may also depend on the physical conditions described by the statistical average with the density matrix $\hat{\rho}$. Since we are interested in non-equilibrium time-evolution of the system where the physical particle mass may not have a definite meaning in the intermediate states, we consider here μ just as a parameter to be chosen at our discretion, for an appropriate choice of the initial conditions specified by the Gaussian density matrix. A different choice of this mass parameter would give different definition for “particle excitations”; the Wigner functions thus depend on the particular choice of the mass parameter.

Suppose we take a different particle mass μ' to define the particle creation and annihilation operators

$$b_{\mathbf{p}}(t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}'}} [\omega_{\mathbf{p}}' \tilde{\phi}_{\mathbf{p}}(t) + i\tilde{\pi}_{-\mathbf{p}}(t)] \quad (46)$$

$$b_{\mathbf{p}}^\dagger(t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}'}} [\omega_{\mathbf{p}}' \tilde{\phi}_{\mathbf{p}}(t) - i\tilde{\pi}_{-\mathbf{p}}(t)] \quad (47)$$

with

$$\omega_{\mathbf{p}}' = \sqrt{p^2 + \mu'^2} \quad (48)$$

These new particle creation and annihilation operators should also obey the commutation relations,

$$[b_{\mathbf{p}}(t), b_{\mathbf{p}'}^\dagger(t)] = \delta_{\mathbf{p}, \mathbf{p}'} \quad (49)$$

$$[b_{\mathbf{p}}(t), b_{\mathbf{p}'}(t)] = [b_{\mathbf{p}}^\dagger(t), b_{\mathbf{p}'}^\dagger(t)] = 0. \quad (50)$$

so that they are related to the original ones by the Bogoliubov transformation:

$$b_{\mathbf{p}} = \cosh \alpha_p a_{\mathbf{p}} + \sinh \alpha_p a_{-\mathbf{p}}^\dagger \quad (51)$$

$$b_{-\mathbf{p}}^\dagger = \sinh \alpha_p a_{\mathbf{p}} + \cosh \alpha_p a_{-\mathbf{p}}^\dagger \quad (52)$$

where the real parameter α_p is determined by requiring that they describe the same fields:

$$\tilde{\phi}(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} \cdot [a_{\mathbf{p}}(t) + a_{-\mathbf{p}}^\dagger(t)] = \sum_{\mathbf{p}} \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{\sqrt{2\omega'_{\mathbf{p}}}} \cdot [b_{\mathbf{p}}(t) + b_{-\mathbf{p}}^\dagger(t)] \quad (53)$$

and this gives

$$e^{\alpha_p} = \sqrt{\frac{\omega'_p}{\omega_p}} \quad (54)$$

or

$$\alpha_p = \frac{1}{2} \log \left(\frac{\omega'_p}{\omega_p} \right) = \frac{1}{4} \log \left(\frac{p^2 + \mu'^2}{p^2 + \mu^2} \right) \quad (55)$$

The new Wigner functions defined by replacing the creation and annihilation by the new ones are related to the original Wigner functions by:

$$\begin{aligned} \mathbf{W}'(\mathbf{p}, \mathbf{k}, t) &= \begin{pmatrix} F'(\mathbf{p}, \mathbf{k}, t) & \bar{G}'(\mathbf{p}, \mathbf{k}, t) \\ G'(\mathbf{p}, \mathbf{k}, t) & \bar{F}'(\mathbf{p}, \mathbf{k}, t) \end{pmatrix} \\ &= \mathbf{M}(\mathbf{p} + \mathbf{k}/2) \mathbf{W}(\mathbf{p}, \mathbf{k}, t) \mathbf{M}(\mathbf{p} - \mathbf{k}/2) \end{aligned} \quad (56)$$

where

$$\mathbf{M}(\mathbf{p}) = \begin{pmatrix} \cosh \alpha_{\mathbf{p}} & \sinh \alpha_{\mathbf{p}} \\ \sinh \alpha_{\mathbf{p}} & \cosh \alpha_{\mathbf{p}} \end{pmatrix} = e^{\alpha_p \tau_1} \quad (57)$$

with

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (58)$$

For a small change of the mass parameter $\mu \rightarrow \mu + \delta\mu$ the Wigner function will change by

$$\begin{aligned} \delta \mathbf{W}(\mathbf{p}, \mathbf{k}, t) &= \tau_1 \mathbf{W}(\mathbf{p}, \mathbf{k}, t) \delta \alpha_{\mathbf{p}+\mathbf{k}/2} + \mathbf{W}(\mathbf{p}, \mathbf{k}, t) \tau_1 \delta \alpha_{\mathbf{p}-\mathbf{k}/2} \\ &= \left(\frac{\tau_1 \mathbf{W}(\mathbf{p}, \mathbf{k}, t)}{4((\mathbf{p} + \mathbf{k}/2)^2 + \mu^2)} + \frac{\mathbf{W}(\mathbf{p}, \mathbf{k}, t) \tau_1}{4((\mathbf{p} - \mathbf{k}/2)^2 + \mu^2)} \right) \mu \delta \mu \end{aligned} \quad (59)$$

In particular,

$$\delta F(\mathbf{p}, \mathbf{k}, t) = \left(\frac{\bar{G}(\mathbf{p}, \mathbf{k}, t)}{4((\mathbf{p} + \mathbf{k}/2)^2 + \mu^2)} + \frac{G(\mathbf{p}, \mathbf{k}, t)}{4((\mathbf{p} - \mathbf{k}/2)^2 + \mu^2)} \right) \mu \delta \mu \quad (60)$$

In uniform equilibrium system the mass parameter may be chosen to "diagonalize" the one-body mean field Hamiltonian. As will be shown later, this procedure will lead to the well-known gap equation, a self-consistency condition to determine μ . If the system is slowly changing in time, one may still use such procedures adjusted to slowly varying quasi-equilibrium conditions, introducing a time dependent effective mass as a dynamical parameter to describe such adiabatic process. For such calculations, the relation (60) may be used to describe the adiabatic change of the Wigner functions by the change of the mass parameter.

In more general non-equilibrium situations as we expect to encounter at the freeze-out stage of expanding matter, however, there may be no such appropriate condition to determine the mass parameter. The situation could even be worse: if the system goes through unstable state with respect to small fluctuation of the field, then the adiabatically determined mass parameter would become pure imaginary, reflecting the extremum, instead of the minimum, of the effective potential. In such case we may keep the value of μ at a certain real value reflecting initial conditions. The instability would then show up as appearance of a growing solution to our kinetic equations; which would eventually be stabilized by the non-linear interaction.

To extract the physical information, such as the particle distribution in the final asymptotic state, we should use the Wigner function defined with the physical particle mass in the vacuum. But these asymptotic physical Wigner

functions may be calculated from the Wigner functions with a different choice of the mass parameter by the relation (56).

2.4 Equation of motion in the mean field approximation

The equation of motion of the classical mean field $\phi_c(\mathbf{r}, t)$ is obtained by taking the quantum statistical average of the field equation (7). With the Gaussian Ansatz for the density matrix, we find

$$\square\phi_c(\mathbf{r}, t) + m^2\phi_c(\mathbf{r}, t) = -\frac{1}{3!}\lambda \left[\phi_c^3(\mathbf{r}, t) + 3\langle\tilde{\phi}^2(\mathbf{r}, t)\rangle\phi_c(\mathbf{r}, t) \right] \quad (61)$$

This equation corresponds to the non-linear Schrödinger equation (also called the Gross-Pitaevskii equation [14]) in the theory of Bose-Einstein condensates. So we may call this equation *non-linear Klein-Gordon equation*. The non-linearity arises due to the self-interaction of the classical field $\phi_c(\mathbf{r}, t)$ (condensate) and also due to the interaction with fluctuations $\langle\tilde{\phi}^2(\mathbf{r}, t)\rangle$ which also depends on $\phi_c(\mathbf{r}, t)$ implicitly. The latter may be interpreted as due to “particle excitations”, since the fluctuation can be expressed by the Wigner functions as

$$\begin{aligned} \langle\tilde{\phi}^2(\mathbf{r}, t)\rangle &= \sum_{\mathbf{p}, \mathbf{p}'} \frac{e^{i(-\mathbf{p}+\mathbf{p}')\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}\sqrt{2\omega_{\mathbf{p}'}}} \langle (a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger)(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger) \rangle \\ &= \sum_{\mathbf{p}, \mathbf{p}'} \frac{e^{i(-\mathbf{p}+\mathbf{p}')\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}\sqrt{2\omega_{\mathbf{p}'}}} \left[F\left(\frac{\mathbf{p}+\mathbf{p}'}{2}, \mathbf{p}-\mathbf{p}', t\right) + \bar{F}\left(\frac{\mathbf{p}+\mathbf{p}'}{2}, \mathbf{p}-\mathbf{p}', t\right) \right. \\ &\quad \left. + G\left(\frac{\mathbf{p}+\mathbf{p}'}{2}, \mathbf{p}-\mathbf{p}', t\right) + \bar{G}\left(\frac{\mathbf{p}+\mathbf{p}'}{2}, \mathbf{p}-\mathbf{p}', t\right) \right] \quad (62) \end{aligned}$$

The time-evolution of the classical mean field $\phi_c(\mathbf{r}, t)$ is thus coupled with the time-evolution of the Wigner functions.

To derive the equation of motion of the Wigner functions, we need to compute the time-derivative of the bilinear forms of the operators $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^\dagger(t)$ which in turn requires computation of the commutators of these operators with the hamiltonian. We decompose the original hamiltonian as

$$H = H_0 + H_1 + H_2 + H_3 + H_4 \quad (63)$$

where H_0 is the classical hamiltonian obtained from H by replacing the quantum fields by their classical expectation values and H_i contain the i -th power of the quantum fluctuation $\hat{\phi}$ (or $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$). A straightforward calculation yields

$$H_1 = \int d\mathbf{r} \left[\pi_c \tilde{\pi} + \nabla \phi_c \nabla \tilde{\phi} + m^2 \phi_c \tilde{\phi} + \frac{\lambda}{3!} \phi_c^3 \tilde{\phi} \right] \quad (64)$$

$$H_2 = \int d\mathbf{r} \left[\frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{1}{2} m^2 \tilde{\phi}^2(\mathbf{r}, t) + \frac{\lambda}{4} \phi_c^2(\mathbf{r}, t) \tilde{\phi}^2(\mathbf{r}, t) \right] \quad (65)$$

$$H_3 = \frac{\lambda}{3!} \int d\mathbf{r} \phi_c(\mathbf{r}, t) \tilde{\phi}^3(\mathbf{r}, t) \quad (66)$$

$$H_4 = \frac{\lambda}{4!} \int d\mathbf{r} \tilde{\phi}^4(\mathbf{r}, t) \quad (67)$$

The commutators of bilinear forms of $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^\dagger(t)$ with H_1 vanish and the commutators with H_3 would give either a linear term or the third power of the fluctuation, both of which may vanish when taking the average with the Gaussian density matrix. What remain to be computed are then the commutators with H_2 and with H_4 . They will give either the bilinear form of $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^\dagger$ or the fourth power of the fluctuations. The Gaussian average of the resultant equations of motion of the bilinear field operators would give the desired equations of motion of the Wigner functions. Details of this computation is given in Appendix A.

The resultant equation of motion of the Wigner functions may be obtained more easily by introducing the mean field Hamiltonian defined by

$$\begin{aligned} H_{\text{mf}} &= \int d\mathbf{r} \left[\frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{2} \Pi(\mathbf{r}, t) \tilde{\phi}^2 \right] \\ &= \int d\mathbf{r} \left[\frac{1}{2} \tilde{\pi}^2 + \frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{1}{2} \mu^2 \tilde{\phi}^2 + \frac{1}{2} \Delta \Pi(\mathbf{r}, t) \tilde{\phi}^2 \right] \end{aligned} \quad (68)$$

where

$$\Pi(\mathbf{r}, t) = \frac{\lambda}{2} \left(\phi_c^2(\mathbf{r}, t) + \langle \tilde{\phi}^2(\mathbf{r}, t) \rangle \right) \quad (69)$$

and

$$\Delta \Pi(\mathbf{r}, t) = \Pi(\mathbf{r}, t) + m^2 - \mu^2 \quad (70)$$

In the momentum representation this mean field Hamiltonian may be written as

$$H_{\text{mf}} = \sum_{\mathbf{p}} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}, \mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{-\mathbf{p}-\mathbf{q}} + a_{\mathbf{p}+\mathbf{q}}^\dagger)}{\sqrt{2\omega_{\mathbf{p}}} \sqrt{2\omega_{\mathbf{p}+\mathbf{q}}}} \quad (71)$$

where

$$\Delta\Pi_{\mathbf{q}}(t) = \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} (\Pi(\mathbf{r}, t) + m^2 - \mu^2) = \Pi_{\mathbf{q}}(t) + (m^2 - \mu^2)\delta_{\mathbf{q},0} \quad (72)$$

The commutator of a bilinear operator product of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ with this mean-field Hamiltonian is given by

$$\begin{aligned} [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_{\text{mf}}] = & -(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} - \sum_{\mathbf{q}} \Delta\Pi_{\mathbf{q}} \cdot \frac{(a_{-\mathbf{p}_1-\mathbf{q}} + a_{\mathbf{p}_1+\mathbf{q}}^\dagger) a_{\mathbf{p}_2}}{\sqrt{2\omega_{\mathbf{p}_1+\mathbf{q}}} \sqrt{2\omega_{\mathbf{p}_2}}} \\ & + \sum_{\mathbf{q}} \Delta\Pi_{\mathbf{q}} \cdot \frac{a_{\mathbf{p}_1}^\dagger (a_{\mathbf{p}_2-\mathbf{q}} + a_{-\mathbf{p}_2+\mathbf{q}}^\dagger)}{\sqrt{2\omega_{\mathbf{p}_1}} \sqrt{2\omega_{\mathbf{p}_2-\mathbf{q}}}} \end{aligned} \quad (73)$$

We show in Appendix that the quantum statistical average of this commutator with the Gaussian density matrix gives precisely the same result for the same statistical average of the commutator with the original Hamiltonian:

$$\langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_{\text{mf}}] \rangle = \langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H] \rangle \quad (74)$$

Therefore one can compute the equations of motion of the Wigner functions using this effective Hamiltonian,

$$i \frac{\partial}{\partial t} F(\mathbf{p}, \mathbf{k}, t) = \langle [a_{\mathbf{p}+\mathbf{k}/2}^\dagger a_{\mathbf{p}-\mathbf{k}/2}, H] \rangle = \langle [a_{\mathbf{p}+\mathbf{k}/2}^\dagger a_{\mathbf{p}-\mathbf{k}/2}, H_{\text{mf}}] \rangle \quad (75)$$

Using this we find,

$$\begin{aligned} i \frac{\partial}{\partial t} F(\mathbf{p}, \mathbf{k}, t) = & -(\omega_{\mathbf{p}+\mathbf{k}/2} - \omega_{\mathbf{p}-\mathbf{k}/2}) F(\mathbf{p}, \mathbf{k}, t) \\ & - \sum_{\mathbf{q}} \Delta\Pi_{\mathbf{q}} \cdot \frac{F(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + G(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \\ & + \sum_{\mathbf{q}} \Delta\Pi_{\mathbf{q}} \cdot \frac{F(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + \bar{G}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}} \end{aligned} \quad (76)$$

Equations of motion of other three Wigner functions $G(\mathbf{p}, \mathbf{k}, t)$, $\bar{G}(\mathbf{p}, \mathbf{k}, t)$, $\bar{F}(\mathbf{p}, \mathbf{k}, t)$ can be also computed from the commutation relations of the product operators $a_{\mathbf{p}_1} a_{\mathbf{p}_2}$, $a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger$, $a_{\mathbf{p}_1} a_{\mathbf{p}_2}^\dagger$, with the mean field Hamiltonian H_{mf} , respectively: We obtain

$$i \frac{\partial}{\partial t} G(\mathbf{p}, \mathbf{k}, t) = (\omega_{\mathbf{p}+\mathbf{k}/2} + \omega_{\mathbf{p}-\mathbf{k}/2}) G(\mathbf{p}, \mathbf{k}, t)$$

$$\begin{aligned}
& + \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{G(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + F(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \\
& + \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{G(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + \bar{F}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}}
\end{aligned} \tag{77}$$

$$\begin{aligned}
i \frac{\partial}{\partial t} \bar{G}(\mathbf{p}, \mathbf{k}, t) &= -(\omega_{\mathbf{p}+\mathbf{k}/2} + \omega_{\mathbf{p}-\mathbf{k}/2}) \bar{G}(\mathbf{p}, \mathbf{k}, t) \\
& - \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\bar{G}(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + \bar{F}(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \\
& - \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\bar{G}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + F(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}}
\end{aligned} \tag{78}$$

and the equation of motion of $\bar{F}(\mathbf{p}, \mathbf{k}, t)$ can be obtained from (76) by the substitution $\mathbf{p} \rightarrow -\mathbf{p}$:

$$\begin{aligned}
i \frac{\partial}{\partial t} \bar{F}(\mathbf{p}, \mathbf{k}, t) &= (\omega_{\mathbf{p}+\mathbf{k}/2} - \omega_{\mathbf{p}-\mathbf{k}/2}) \bar{F}(\mathbf{p}, \mathbf{k}, t) \\
& + \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\bar{F}(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + \bar{G}(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \\
& - \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\bar{F}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) + G(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}}
\end{aligned} \tag{79}$$

These equations form a closed system of coupled differential equations with the non-linear Klein-Gordon equation (61) which may be rewritten as

$$\Box \phi_c(\mathbf{r}, t) + (\mu^2 + \Delta \Pi(\mathbf{r}, t)) \phi_c(\mathbf{r}, t) = -\frac{\lambda}{3} \phi_c^3(\mathbf{r}, t). \tag{80}$$

These four equations of motion of the Wigner function may be combined into a single matrix form as

$$\begin{aligned}
i \frac{\partial}{\partial t} \mathbf{W}(\mathbf{p}, \mathbf{k}, t) &= -\omega_{\mathbf{p}+\mathbf{k}/2} \tau_3 \mathbf{W}(\mathbf{p}, \mathbf{k}, t) + \omega_{\mathbf{p}-\mathbf{k}/2} \mathbf{W}(\mathbf{p}, \mathbf{k}, t) \tau_3 \\
& - \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\tau_3(1 + \tau_1) \mathbf{W}(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \\
& + \sum_{\mathbf{q}} \Delta \Pi_{\mathbf{q}} \cdot \frac{\mathbf{W}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t)(1 + \tau_1) \tau_3}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}} \sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}}
\end{aligned}$$

(81)

3 Uniform system

For a uniform system, we expect that the classical mean field and the self-energy become functions only of time:

$$\phi_c(\mathbf{r}, t) = \phi_0(t), \quad \Pi(\mathbf{r}, t) = \Pi_0(t) \quad (82)$$

Thus the non-linear Klein-Gordon equation (80) becomes

$$\ddot{\phi}_0(\mathbf{r}, t) + (\mu^2 + \Delta\Pi_0(t))\phi_0(t) = -\frac{\lambda}{3}\phi_0^3(t). \quad (83)$$

and the mean-field Hamiltonian is reduced to

$$H_{\text{mf}} = \sum_{\mathbf{p}} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \Delta\Pi_0(t) \sum_{\mathbf{p}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger) \quad (84)$$

In this case the Wigner functions contain non-vanishing components only for the diagonal elements ($\mathbf{k} = 0$) so that they may be written as

$$F(\mathbf{p}, \mathbf{k}, t) = F_0(\mathbf{p}, t)\delta(\mathbf{k}) \quad (85)$$

$$G(\mathbf{p}, \mathbf{k}, t) = G_0(\mathbf{p}, t)\delta(\mathbf{k}) \quad (86)$$

$$\bar{G}(\mathbf{p}, \mathbf{k}, t) = \bar{G}_0(\mathbf{p}, t)\delta(\mathbf{k}) \quad (87)$$

$$\bar{F}(\mathbf{p}, \mathbf{k}, t) = (F_0(-\mathbf{p}, t) + 1)\delta_{\mathbf{k},0} \quad (88)$$

Then the equations of motion of the Wigner functions become

$$i\frac{\partial}{\partial t}F_0(\mathbf{p}, t) = \frac{\Delta\Pi_0}{2\omega_{\mathbf{p}}} (\bar{G}_0(\mathbf{p}, t) - G_0(\mathbf{p}, t)) \quad (89)$$

$$i\frac{\partial}{\partial t}\bar{F}_0(\mathbf{p}, t) = \frac{\Delta\Pi_0}{2\omega_{\mathbf{p}}} (\bar{G}_0(\mathbf{p}, t) - G_0(\mathbf{p}, t)) \quad (90)$$

$$\begin{aligned} i\frac{\partial}{\partial t}G_0(\mathbf{p}, t) = & \left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right) G_0(\mathbf{p}, t) \\ & + \frac{\Delta\Pi_0}{2\omega_{\mathbf{p}}} (F_0(\mathbf{p}, t) + \bar{F}_0(\mathbf{p}, t)) \end{aligned} \quad (91)$$

$$i\frac{\partial}{\partial t}\bar{G}_0(\mathbf{p}, t) = -\left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right) \bar{G}_0(\mathbf{p}, t)$$

$$-\frac{\Delta\Pi_0}{2\omega_{\mathbf{p}}} \left(F_0(\mathbf{p}, t) + \bar{F}_0(\mathbf{p}, t) \right) \quad (92)$$

These coupled equations have a time-independent solution of the form

$$G_0(\mathbf{p}) = \bar{G}_0(\mathbf{p}) = c \left(F_0(\mathbf{p}) + \bar{F}_0(\mathbf{p}) \right) \quad (93)$$

with

$$c = -\frac{\Delta\Pi_0}{\omega_{\mathbf{p}}} \left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}} \right)^{-1} \quad (94)$$

Thus if we define the particle mass μ so as to satisfy $\Delta\Pi_0 = 0$, then two components of the Wigner functions $G_0(\mathbf{p})$ and $\bar{G}_0(\mathbf{p})$ vanish for all \mathbf{p} . In this case, $F_0(\mathbf{p})$ may be interpreted as the momentum distribution of physical particle excitations.

To find time-dependent solutions, we write

$$F_{\pm}(\mathbf{p}, t) = F_0(\mathbf{p}, t) \pm \bar{F}_0(\mathbf{p}, t) \quad (95)$$

$$G_{\pm}(\mathbf{p}, t) = G_0(\mathbf{p}, t) \pm \bar{G}_0(\mathbf{p}, t) \quad (96)$$

and rewrite the equations as

$$i\frac{\partial}{\partial t}F_+(\mathbf{p}, t) = -\frac{\Delta\Pi_0(t)}{\omega_{\mathbf{p}}}G_-(\mathbf{p}, t) \quad (97)$$

$$i\frac{\partial}{\partial t}F_-(\mathbf{p}, t) = 0 \quad (98)$$

$$i\frac{\partial}{\partial t}G_+(\mathbf{p}, t) = \left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0(t)}{\omega_{\mathbf{p}}} \right) G_-(\mathbf{p}, t) \quad (99)$$

$$i\frac{\partial}{\partial t}G_-(\mathbf{p}, t) = \left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0(t)}{\omega_{\mathbf{p}}} \right) G_+(\mathbf{p}, t) + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}F_+(\mathbf{p}, t) \quad (100)$$

These equations are to be solved together with the non-linear Klein-Gordon equation (83) with

$$\Delta\Pi_0(t) = \frac{\lambda}{2} \left[\phi_0^2(t) + \sum_{\mathbf{p}} \frac{F_+(\mathbf{p}, t) + G_+(\mathbf{p}, t)}{2\omega_{\mathbf{p}}} + m^2 - \mu^2 \right] \quad (101)$$

From these equations we can derive second order differential equation:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}F_+(\mathbf{p}, t) = & -\left(\frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right)^2 F_+(\mathbf{p}, t) - \frac{\Delta\Pi_0}{E_{\mathbf{p}}}\left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right) G_+(\mathbf{p}, t) \\ & + i\frac{1}{\omega_{\mathbf{p}}}\frac{d\Delta\Pi_0(t)}{dt}G_-(\mathbf{p}, t)\end{aligned}\quad (102)$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}G_+(\mathbf{p}, t) = & -\frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right) F_+(\mathbf{p}, t) - \left(2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}\right)^2 G_+(\mathbf{p}, t) \\ & - i\frac{1}{\omega_{\mathbf{p}}}\frac{d\Delta\Pi_0(t)}{dt}G_-(\mathbf{p}, t)\end{aligned}\quad (103)$$

with

$$\frac{d\Delta\Pi_0(t)}{dt} = \lambda \left[\phi_0(t)\dot{\phi}_0(t) + \sum_{\mathbf{p}} G_-(\mathbf{p}, t) \right] \quad (104)$$

We observe that each of these coupled equations looks like an equation of a forced oscillator. For example the frequency of the oscillator F_+ is given by $\frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}$ while the frequency of the oscillator G_+ is given by $2\omega_{\mathbf{p}} + \frac{\Delta\Pi_0}{\omega_{\mathbf{p}}}$ so that G_+ oscillates much rapidly than F_+ . The coupling of these oscillators may produce an interesting effects which may be studied by numerical integration.

4 Statistical equilibrium

In statistical equilibrium, we have the Bose distribution

$$F_0(\mathbf{p}) = f_{\text{eq.}}(\mathbf{p}) = \frac{1}{e^{\omega_{\mathbf{p}}\beta} - 1} \quad (105)$$

together with

$$G_0(\mathbf{p}) = g_{\text{eq.}}(\mathbf{p}) = 0. \quad (106)$$

where $\beta = 1/k_B T$ is the inverse temperature. The condition of vanishing $\Delta\Pi_0$ implies from (72)

$$\frac{\lambda}{2} \left(\phi_0^2 + \langle \tilde{\phi}^2 \rangle_{\text{eq.}} \right) + m^2 - \mu^2 = 0. \quad (107)$$

where the thermal fluctuation of the quantum field is given by the relation (62) as

$$\langle \tilde{\phi}^2 \rangle_{\text{eq.}} = \sum_{\mathbf{p}} \frac{1}{\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \quad (108)$$

On the other hand, the condensate amplitude ϕ_0 in equilibrium should also satisfy the static non-linear Klein-Gordon equation,

$$m^2 \phi_0 = -\frac{1}{3!} \lambda \left[\phi_0^3 + 3 \langle \tilde{\phi}^2 \rangle_{\text{eq.}} \phi_0 \right] \quad (109)$$

From these two conditions, we find, for $m^2 < 0$,

$$\mu^2 = -2m^2 - \lambda \sum_{\mathbf{p}} \frac{1}{\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \quad (110)$$

and

$$\lambda \phi_0^2 = 3\mu^2 \quad (111)$$

We note that, since the thermal distribution $f_{\text{eq.}}$ depends on the mass (gap) parameter μ through $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \mu^2}$, the equation (110) determines the mass parameter μ self-consistently as a function of the temperature. This equation is called the *gap equation* [26, 27]. As the temperature increases the mass gap μ and the condensate amplitude ϕ_0 decreases and vanishes at high temperature.

To find the behavior of the mass parameter as a function of the temperature we can use the following formula [26, 34] for the integral of the bose distribution function:

$$\sum_{\mathbf{p}} \frac{1}{\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{\omega_{\mathbf{p}}} \frac{1}{e^{\omega_{\mathbf{p}}\beta} - 1} = \frac{\beta^{-2}}{2\pi^2} I_-^{(2)}(\mu\beta) \quad (112)$$

where the dimensionless function $I_-^{(2)}(x)$ is given as

$$\begin{aligned} I_-^{(2)}(x) &\equiv \int_0^\infty \frac{k^2 dk}{\sqrt{k^2 + x^2}} \frac{1}{e^{\sqrt{k^2 + x^2}} - 1} \\ &= \frac{\pi^2}{6} - \frac{\pi}{2}x - \frac{1}{4}x^2 \ln \frac{x}{4\pi} + \left(\frac{1}{8} - \frac{1}{4}\gamma \right) x^2 - \frac{\zeta(3)}{32\pi} x^4 + \mathcal{O}(x^6) \end{aligned} \quad (113)$$

($\gamma = 0.57721 \dots$ is Euler's number) . The gap equation (110) has a solution $\mu = 0$ at T_c determined by

$$(k_B T_c)^2 = -\frac{24}{\lambda} m^2 \quad (114)$$

However the solution exhibits the behavior of the first order transition due to the non-analytic behavior of the function $I_-^{(2)}(x)$ [34]. This is a generic feature of the mean field approximation [27] which may be a theoretical artifact and may not survive the inclusion of correlations missing in the mean field approximation. An improvement of the mean field approximation has been proposed in [35] and later it has been shown to lead to the second order transition in two-loop approximations [36].

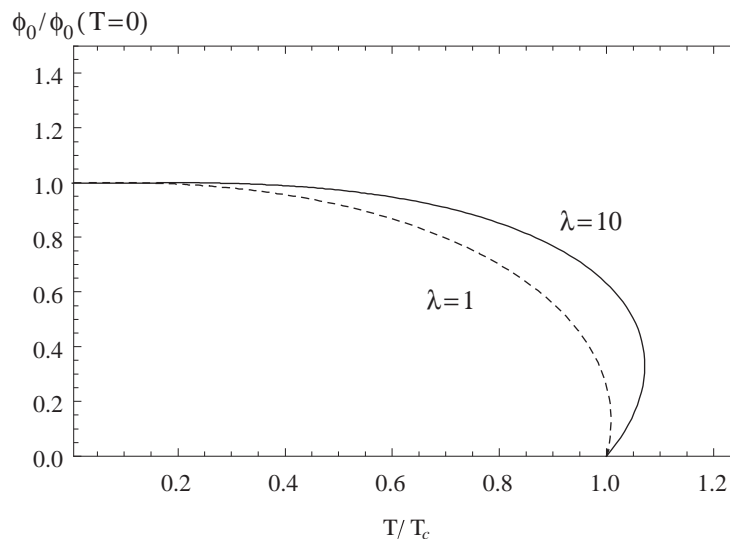


Fig. 1. Temperature dependence of the mass parameter μ for $\lambda = 1$ (dashed line) and $\lambda = 10$ (solid line)

In Fig. 1 we plot ϕ_0 as determined by (109) as a function of temperature. It shows a behavior characteristic of the first order phase transition: there is a region $T_c < T < T_0$ where there are three solutions of (109), one at $\phi_0 = 0$ and other two at $\phi = \phi_1 \neq 0$ as well as $\phi = \phi_0 \neq 0$. One expects that two solutions $\phi = \phi_0$ and $\phi = \phi_0$ correspond to two local minima of the effective potential $V(\phi)$ while the other one $\phi = \phi_1$ corresponds to the local maximum of $V(\phi)$. Indeed we can show this explicitly by constructing the effective potential $V(\phi)$ by demanding that the local extremal condition $\partial V(\phi)/\partial \phi = 0$ coincides with the condition (109).

$$V(\phi) = \int d\phi \left(m^2 \phi + \frac{\lambda}{3!} \phi^3 + \frac{\lambda}{2} \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \phi \right) \quad (115)$$

The integration of the first two terms are trivial. Beside irrelevant integration constant, which we may choose to be zero, we obtain

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda}{2} \int d\phi \phi \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \quad (116)$$

To carry out the remaining integral over ϕ we demand that the mass parameter μ depends on ϕ through the relation

$$\mu^2 = m^2 + \frac{\lambda}{2} (\phi^2 + \langle \phi^2 \rangle_{\text{eq.}}) = m^2 + \frac{\lambda}{2} \left(\phi^2 + \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \right) \quad (117)$$

which appears to be equivalent to the condition (107), but we assume that this relation holds not only for the equilibrium value of ϕ , namely at $\phi = \phi_0$, but also for any value of ϕ . Noting

$$\frac{\lambda}{2} d\phi \phi = \left(1 - \frac{\lambda}{2} \frac{\partial}{\partial \mu^2} \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \right) d\mu \mu \quad (118)$$

which results from (117), we transform the integration variable from ϕ to μ^2 :

$$\begin{aligned} \frac{\lambda}{2} \int d\phi \phi \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} &= \frac{1}{2} \int d\mu^2 \left(1 - \frac{\lambda}{2} \frac{\partial}{\partial \mu^2} \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \right) \sum_{\mathbf{p}'} \frac{f_{\text{eq.}}(\mathbf{p}')}{\omega_{\mathbf{p}'}} \\ &= \frac{1}{2} \int d\mu^2 \sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} - \frac{\lambda}{8} \left(\sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \right)^2 + \text{const.} \end{aligned} \quad (119)$$

Remaining integral over μ^2 can be carried out as

$$\frac{1}{2} \sum_{\mathbf{p}} \int d\mu^2 \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} = \frac{1}{2\beta} \sum_{\mathbf{p}} \ln [1 - \exp(-\beta\omega_{\mathbf{p}})] = - \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{3\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \quad (120)$$

where in deriving the final expression we performed integration by part in p and omitted again the irrelevant integration constant.

To sum up, we find the following expression for the effective potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{3\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) - \frac{\lambda}{8} \left(\sum_{\mathbf{p}} \frac{f_{\text{eq.}}(\mathbf{p})}{\omega_{\mathbf{p}}} \right)^2 + \text{const.} \quad (121)$$

where ϕ dependence of the last two terms are implicitly given from the ϕ -dependence of μ . We note that this result coincides, in the neglect of the

renormalization effects due to the meson mass shift, with the effective potential obtained by Amelino-Camelia and Pi [37] using the Cornwall-Jackiw-Tombolis (CJT) composite operator effective potential formalism [38]. We note that our mass parameter μ corresponds to the variational mass parameter M of the CJT potential. The third term may be interpreted as the pressure of an ideal gas of the quasi-particles obeying the dispersion $\varepsilon = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \mu^2}$.

The effective potential given by the formula (121) takes in general a complex value since μ^2 becomes negative at low temperature and at small value of ϕ . At low temperatures the potential develops a second local minimum at non-zero value of ϕ , which becomes the absolute minimum below certain temperature T_1 ($T_c < T_1 < T_0$) where the first order transition takes place in equilibrium and the order parameter ϕ_0 jumps discontinuously from 0 to ϕ_0 as the temperature is lowered.

5 Slowly varying system: the Vlasov equation

In the presence of inhomogeneity the Wigner functions acquire non-vanishing elements with $\mathbf{k} \neq 0$. If we assume that this inhomogeneity are due to the long-wavelength fluctuations in the system,

$$k, q \ll p \quad (122)$$

In this case we can obtain a familiar form of the Vlasov equation [39] from the equation of motion of the Wigner function by the procedure usually referred to as the gradient expansion [40].

To show this we make the following approximations:

$$\omega_{\mathbf{p}+\mathbf{k}/2} + \omega_{\mathbf{p}-\mathbf{k}/2} \simeq 2\omega_{\mathbf{p}} \quad (123)$$

$$\omega_{\mathbf{p}+\mathbf{k}/2} - \omega_{\mathbf{p}-\mathbf{k}/2} \simeq \frac{\mathbf{p} \cdot \mathbf{k}}{\omega_{\mathbf{p}}} \quad (124)$$

and

$$\frac{1}{\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2}}\sqrt{2\omega_{\mathbf{p}+\mathbf{k}/2+\mathbf{q}}}} \simeq \frac{1}{2\omega_{\mathbf{p}}} \left(1 - \frac{\mathbf{p} \cdot (\mathbf{k} + \mathbf{q})}{\omega_{\mathbf{p}}^2} \right) \quad (125)$$

$$\frac{1}{\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2}}\sqrt{2\omega_{\mathbf{p}-\mathbf{k}/2-\mathbf{q}}}} \simeq \frac{1}{2\omega_{\mathbf{p}}} \left(1 + \frac{\mathbf{p} \cdot (\mathbf{k} + \mathbf{q})}{\omega_{\mathbf{p}}^2} \right) \quad (126)$$

We also make use of the Taylor expansion of the Wigner functions:

$$F(\mathbf{p} \pm \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) \simeq F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \pm \frac{1}{2} \mathbf{q} \cdot \nabla_{\mathbf{p}} F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \quad (127)$$

$$G(\mathbf{p} + \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) \simeq G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) + \frac{1}{2} \mathbf{q} \cdot \nabla_{\mathbf{p}} G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \quad (128)$$

$$\bar{G}(\mathbf{p} - \mathbf{q}/2, \mathbf{k} + \mathbf{q}, t) \simeq \bar{G}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) - \frac{1}{2} \mathbf{q} \cdot \nabla_{\mathbf{p}} \bar{G}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \quad (129)$$

Then the equations of motion (76) of the Wigner function $F(\mathbf{p}, \mathbf{k}, t)$ becomes

$$\begin{aligned} \frac{\partial}{\partial t} F(\mathbf{p}, \mathbf{k}, t) = & i \frac{\mathbf{p} \cdot \mathbf{k}}{\omega_{\mathbf{p}}} F(\mathbf{p}, \mathbf{k}, t) + i \sum_{\mathbf{q}} \frac{\Delta \Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} \mathbf{q} \cdot \nabla_{\mathbf{p}} F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \\ & - i \sum_{\mathbf{q}} \frac{\Delta \Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} \frac{\mathbf{p} \cdot (\mathbf{k} + \mathbf{q})}{\omega_{\mathbf{p}}^2} F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \\ & + i \sum_{\mathbf{q}} \frac{\Delta \Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} (G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) - \bar{G}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t)) \\ & - i \sum_{\mathbf{q}} \frac{\Delta \Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} \frac{\mathbf{p} \cdot (\mathbf{k} + \mathbf{q})}{\omega_{\mathbf{p}}^2} (G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) + \bar{G}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t)) \\ & + i \sum_{\mathbf{q}} \frac{\Delta \Pi_{\mathbf{q}}}{4\omega_{\mathbf{p}}} \mathbf{q} \cdot \nabla_{\mathbf{p}} (G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) + \bar{G}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t)) \end{aligned} \quad (130)$$

On the right hand side, the first term and the second term are disguised forms of the drift term and the Vlasov term respectively.

To obtain more familiar form we make the Fourier transforms (42) and (43) of the Wigner functions. Then we find

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{p}, \mathbf{r}, t) = & - \frac{\mathbf{p}}{\omega_{\mathbf{p}}} \cdot \nabla_{\mathbf{r}} f(\mathbf{p}, \mathbf{r}, t) + \nabla_{\mathbf{r}} \left(\frac{\Delta \Pi(\mathbf{r}, t)}{2\omega_{\mathbf{p}}} \right) \cdot \nabla_{\mathbf{p}} f(\mathbf{p}, \mathbf{r}, t) \\ & + \frac{\Delta \Pi(\mathbf{r}, t)}{2\omega_{\mathbf{p}}} \frac{\mathbf{p}}{\omega_{\mathbf{p}}^2} \cdot \nabla_{\mathbf{r}} f(\mathbf{p}, \mathbf{r}, t) \\ & + i \frac{\Delta \Pi(\mathbf{r}, t)}{2\omega_{\mathbf{p}}} (g(\mathbf{p}, \mathbf{r}, t) - \bar{g}(\mathbf{p}, \mathbf{r}, t)) \\ & + \frac{\Delta \Pi(\mathbf{r}, t)}{2\omega_{\mathbf{p}}^2} \frac{\mathbf{p}}{\omega_{\mathbf{p}}} \cdot \nabla_{\mathbf{r}} (g(\mathbf{p}, \mathbf{r}, t) + \bar{g}(\mathbf{p}, \mathbf{r}, t)) \\ & + \nabla_{\mathbf{r}} \left(\frac{\Delta \Pi(\mathbf{r}, t)}{4\omega_{\mathbf{p}}} \right) \cdot \nabla_{\mathbf{p}} (g(\mathbf{p}, \mathbf{r}, t) + \bar{g}(\mathbf{p}, \mathbf{r}, t)) \end{aligned} \quad (131)$$

Now it is clear that the first term is the drift term which describes the change of the particle position by the drift with the velocity $\mathbf{v}_{\mathbf{p}} = \mathbf{p}/\omega_{\mathbf{p}}$. The second

term can be interpreted as the Vlasov term which represents the change of particle momentum due to the continuous acceleration by the velocity dependent equivalent potential,

$$U_{\mathbf{p}}(\mathbf{r}, t) = \frac{\Delta\Pi(\mathbf{r}, t)}{2\omega_{\mathbf{p}}} = \frac{\Pi(\mathbf{r}, t) + m^2 - \mu^2}{2\omega_{\mathbf{p}}} \quad (132)$$

acting on the particle. The third term appears to be the correction to the drift term due to the local change of particle mass $\mu \rightarrow \mu' = \mu + \Delta\Pi(\mathbf{r}, t)$ which causes change in particle velocity. Other three terms are associated with the other components of the Wigner function and has no counter parts in non-relativistic Vlasov equation.

Noting that

$$\frac{\partial\omega_{\mathbf{p}}}{\partial\mathbf{p}} = \frac{\mathbf{p}}{\omega_{\mathbf{p}}}, \quad (133)$$

and

$$\frac{\partial U_{\mathbf{p}}(\mathbf{r}, t)}{\partial\mathbf{p}} = -\frac{U_{\mathbf{p}}(\mathbf{r}, t)\mathbf{p}}{\omega_{\mathbf{p}}^2}, \quad (134)$$

the above equation may be rewritten in a more compact form:

$$\begin{aligned} \frac{\partial}{\partial t}f(\mathbf{p}, \mathbf{r}, t) + \nabla_{\mathbf{p}}\varepsilon(\mathbf{p}, \mathbf{r}, t) \cdot \nabla_{\mathbf{r}}f(\mathbf{p}, \mathbf{r}, t) - \nabla_{\mathbf{r}}\varepsilon(\mathbf{p}, \mathbf{r}, t) \cdot \nabla_{\mathbf{p}}f(\mathbf{p}, \mathbf{r}, t) = \\ iU_{\mathbf{p}}(\mathbf{r}, t)g_{-}(\mathbf{p}, \mathbf{r}, t) - \frac{1}{2}\nabla_{\mathbf{p}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}}g_{+}(\mathbf{p}, \mathbf{r}, t) \\ + \frac{1}{2}\nabla_{\mathbf{r}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}}g_{+}(\mathbf{p}, \mathbf{r}, t) \end{aligned} \quad (135)$$

where

$$\varepsilon(\mathbf{p}, \mathbf{r}, t) = \omega_{\mathbf{p}} + U_{\mathbf{p}}(\mathbf{r}, t) \quad (136)$$

and

$$g_{\pm}(\mathbf{p}, \mathbf{r}, t) = g(\mathbf{p}, \mathbf{r}, t) \pm \bar{g}(\mathbf{p}, \mathbf{r}, t). \quad (137)$$

The quantity $\varepsilon(\mathbf{p}, \mathbf{r}, t)$ plays the same role as the quasi-particle energy which appears in the kinetic equation of Landau's Fermi-liquid theory [28].³ We note that because of the relation (45), $g_-(\mathbf{p}, \mathbf{r}, t)$ is a pure imaginary function, while $g_+(\mathbf{p}, \mathbf{r}, t)$, $f(\mathbf{p}, \mathbf{r}, t)$, $U_{\mathbf{p}}(\mathbf{r}, t)$, and $\varepsilon(\mathbf{p}, \mathbf{r}, t)$ are all real functions.

In the long wavelength approximation, the equation of motion of $G(\mathbf{p}, \mathbf{k}, t)$ becomes

$$\begin{aligned} \frac{\partial}{\partial t} G(\mathbf{p}, \mathbf{k}, t) = & -2i\omega_{\mathbf{p}} G(\mathbf{p}, \mathbf{k}, t) - i \sum_{\mathbf{q}} \frac{\Delta\Pi_{\mathbf{q}}}{\omega_{\mathbf{p}}} G(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \\ & - i \sum_{\mathbf{q}} \frac{\Delta\Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} \left(F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) + \bar{F}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \right) \\ & + i \sum_{\mathbf{q}} \frac{\Delta\Pi_{\mathbf{q}}}{2\omega_{\mathbf{p}}} \frac{\mathbf{p} \cdot (\mathbf{k} + \mathbf{q})}{\omega_{\mathbf{p}}^2} \left(F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) - \bar{F}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \right) \\ & - i \sum_{\mathbf{q}} \frac{\Delta\Pi_{\mathbf{q}}}{4\omega_{\mathbf{p}}} \mathbf{q} \cdot \nabla_{\mathbf{p}} \left(F(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) - \bar{F}(\mathbf{p}, \mathbf{k} + \mathbf{q}, t) \right) \end{aligned} \quad (138)$$

which may be rewritten for the Fourier transforms in a compact form as

$$\begin{aligned} \frac{\partial}{\partial t} g(\mathbf{p}, \mathbf{r}, t) + 2i\varepsilon(\mathbf{p}, \mathbf{r}, t)g(\mathbf{p}, \mathbf{r}, t) = & -iU_{\mathbf{p}}(\mathbf{r}, t)f_+(\mathbf{p}, \mathbf{r}, t) \\ & - \frac{1}{2}\nabla_{\mathbf{p}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}}f_-(\mathbf{p}, \mathbf{r}, t) + \frac{1}{2}\nabla_{\mathbf{r}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}}f_-(\mathbf{p}, \mathbf{r}, t) \end{aligned} \quad (139)$$

where $f_{\pm}(\mathbf{p}, \mathbf{r}, t) = f(\mathbf{p}, \mathbf{r}, t) \pm \bar{f}(\mathbf{p}, \mathbf{r}, t)$. Thus the kinetic equation for $g(\mathbf{p}, \mathbf{r}, t)$ does not look like a Vlasov equation; it takes a form of a simple oscillator equation with the oscillator frequency $2\varepsilon(\mathbf{p}, \mathbf{r}, t)$ exposed to the external perturbation created by the particle distribution $f(\mathbf{p}, \mathbf{r}, t)$. Similar equation is derived for $\bar{g}(\mathbf{p}, \mathbf{r}, t)$ which is the complex conjugate of $g(\mathbf{p}, \mathbf{r}, t)$.

Taking the complex conjugate of the Eq. (139), we obtain the kinetic equation for $\bar{g}(\mathbf{p}, \mathbf{r}, t) = g^*(\mathbf{p}, \mathbf{r}, t)$:

$$\begin{aligned} \frac{\partial}{\partial t} \bar{g}(\mathbf{p}, \mathbf{r}, t) - 2i\varepsilon(\mathbf{p}, \mathbf{r}, t)\bar{g}(\mathbf{p}, \mathbf{r}, t) = & -iU_{\mathbf{p}}(\mathbf{r}, t)f_+(\mathbf{p}, \mathbf{r}, t) \\ & - \frac{1}{2}\nabla_{\mathbf{p}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}}f_-(\mathbf{p}, \mathbf{r}, t) + \frac{1}{2}\nabla_{\mathbf{r}}U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}}f_-(\mathbf{p}, \mathbf{r}, t) \end{aligned}$$

³ Relativistic extension of the general framework of the Landau Fermi-liquid theory has been made by Baym and Chin [31] and applied by one (TM) of the present authors [32] to Walecka's relativistic mean field theory of cold dense nuclear matter [33].

(140)

Adding or subtracting (139) and (140), we find for the real function $g_+(\mathbf{p}, \mathbf{r}, t)$ and the pure imaginary function $g_-(\mathbf{p}, \mathbf{r}, t)$

$$\begin{aligned} \frac{\partial}{\partial t} g_+(\mathbf{p}, \mathbf{r}, t) = & 2i\varepsilon(\mathbf{p}, \mathbf{r}, t)g_-(\mathbf{p}, \mathbf{r}, t) - \nabla_{\mathbf{p}} U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} f_-(\mathbf{p}, \mathbf{r}, t) \\ & + \nabla_{\mathbf{r}} U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} f_-(\mathbf{p}, \mathbf{r}, t) \end{aligned} \quad (141)$$

$$\frac{\partial}{\partial t} g_-(\mathbf{p}, \mathbf{r}, t) = 2i\varepsilon(\mathbf{p}, \mathbf{r}, t)g_+(\mathbf{p}, \mathbf{r}, t) - 2iU_{\mathbf{p}}(\mathbf{r}, t)f_+(\mathbf{p}, \mathbf{r}, t) \quad (142)$$

In the long wavelength approximation, we also have

$$\begin{aligned} \langle \hat{\phi}^2(\mathbf{r}, t) \rangle & \simeq \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} [f(\mathbf{p}, \mathbf{r}, t) + \bar{f}(\mathbf{p}, \mathbf{r}, t) + g(\mathbf{p}, \mathbf{r}, t) + \bar{g}(\mathbf{p}, \mathbf{r}, t)] \\ & = \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} [f_+(\mathbf{p}, \mathbf{r}, t) + g_+(\mathbf{p}, \mathbf{r}, t)] \end{aligned} \quad (143)$$

which appears in the non-linear Klein-Gordon equation (61) for the condensate and in the self energy term (69) in the Vlasov equation.

6 Dispersion relation of the excitations near equilibrium

We now apply our coupled kinetic equations, consisting of the non-linear Klein-Gordon equation and the Vlasov equations, to find a dispersion relation of the excitations in the system near equilibrium. The dispersion relations of the excitations of the finite temperature system may be computed using the propagator (two-time) formalism, either with real-times (Schwinger-Keldysh) formalism [42] or with imaginary-times (Matsubara) formalism with subsequent analytic continuation [43]. Although such calculations have been performed by various authors[35, 44, 45] numerical results have been presented focusing on the time-like region of the spectral functions. We study with our kinetic theory the entire range of the (ω, k) plane including the space-like excitations of the system.⁴ As has been shown by one of the present authors [32] the

⁴ We note that [35] contains a general formula of the meson spectral function covering all kinematic ranges including space-like region. Their results are indeed very similar to ours, besides that the divergent vacuum loops are included with an elaborate temperature-dependent renormalization procedure and that the “tree-level” masses are used in the modified loop calculation, while we ignore the vacuum loops

calculation of the zero sound mode in the degenerate Fermi liquid using the relativistic Landau kinetic equation[31], similar to ours, reproduces the same result as the propagator theory in the long wavelength limits. We thus expect that our result may also reproduce the long wavelength behaviors of the excitations obtained from the propagator theory.

For this purpose we assume that the distribution functions consist of uniform equilibrium term and a small deviation from it:

$$f(\mathbf{p}, \mathbf{r}, t) = f_{\text{eq.}}(\mathbf{p}) + \delta f(\mathbf{p}, \mathbf{r}, t) \quad (144)$$

$$g(\mathbf{p}, \mathbf{r}, t) = g_{\text{eq.}}(\mathbf{p}) + \delta g(\mathbf{p}, \mathbf{r}, t) \quad (145)$$

where

$$f_{\text{eq.}}(\mathbf{p}) = \frac{1}{e^{\omega_{\mathbf{p}}\beta} - 1} \quad \text{and} \quad g_{\text{eq.}}(\mathbf{p}) = 0. \quad (146)$$

We also assume the mean meson field consists of uniform equilibrium term and a deviation from it

$$\phi_c(\mathbf{r}, t) = \phi_0 + \delta\phi(\mathbf{r}, t) \quad (147)$$

where ϕ_0 is determined by the static non-linear Klein-Gordon equation (109) together with the solution of the gap equation (110) for the mass gap μ .

Linearization of the non-linear Klein-Gordon equation with respect to $\delta\phi(\mathbf{r}, t)$ yields

$$\square\delta\phi(\mathbf{r}, t) + m^2\delta\phi(\mathbf{r}, t) = -\frac{1}{2}\lambda \left[\phi_0^2\delta\phi(\mathbf{r}, t) + \langle\tilde{\phi}^2\rangle_{\text{eq.}}\delta\phi(\mathbf{r}, t) + \delta\langle\tilde{\phi}^2(\mathbf{r}, t)\rangle\phi_0 \right] \quad (148)$$

where the thermal fluctuation of the quantum field in equilibrium is given by

$$\langle\tilde{\phi}^2\rangle_{\text{eq.}} = \sum_{\mathbf{p}} \frac{1}{\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \quad (149)$$

and the deviation $\delta\langle\tilde{\phi}^2(\mathbf{r}, t)\rangle$ is expressed in term of the small deviation of the distribution function from equilibrium value.

and the mass of our quasi-particle excitations is determined by the gap equation which includes the effect of one-loop diagrams in the language of conventional loop expansion.

$$\begin{aligned}
\delta\langle\hat{\phi}^2(\mathbf{r}, t)\rangle &= \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left[\delta f(\mathbf{p}, \mathbf{r}, t) + \delta\bar{f}(\mathbf{p}, \mathbf{r}, t) + \delta g(\mathbf{p}, \mathbf{r}, t) + \delta\bar{g}(\mathbf{p}, \mathbf{r}, t) \right] \\
&= \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} [\delta f(\mathbf{p}, \mathbf{r}, t) + \delta f(-\mathbf{p}, \mathbf{r}, t) + \delta g(\mathbf{p}, \mathbf{r}, t) + \delta g^*(\mathbf{p}, \mathbf{r}, t)]
\end{aligned} \tag{150}$$

where in deriving the last line we have used

$$\delta\bar{f}(\mathbf{p}, \mathbf{r}, t) = \delta f(-\mathbf{p}, \mathbf{r}, t) \tag{151}$$

$$\delta\bar{g}(\mathbf{p}, \mathbf{r}, t) = \delta g^*(\mathbf{p}, \mathbf{r}, t) \tag{152}$$

which follow from (44) and (45) respectively.

By the linearization with respect to the small increments, $\delta f(\mathbf{p}, \mathbf{r}, t)$, $\delta g(\mathbf{p}, \mathbf{r}, t)$, the Vlasov equations become

$$\frac{\partial}{\partial t} \delta f(\mathbf{p}, \mathbf{r}, t) + \nabla_{\mathbf{p}} \omega_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \delta f(\mathbf{p}, \mathbf{r}, t) - \nabla_{\mathbf{r}} \delta U_{\mathbf{p}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} f_{\text{eq.}}(\mathbf{p}) = 0 \tag{153}$$

$$\frac{\partial}{\partial t} \delta g(\mathbf{p}, \mathbf{r}, t) + 2i\omega_{\mathbf{p}} \delta g(\mathbf{p}, \mathbf{r}, t) = -i2\delta U_{\mathbf{p}}(\mathbf{r}, t) f_{\text{eq.}}(\mathbf{p}) \tag{154}$$

The right hand side of the Vlasov equation (135) vanishes because $\Delta\Pi(\mathbf{p}, \mathbf{r}, t)$ vanishes in equilibrium. The fluctuation in the mean field potential $\delta U_{\mathbf{p}}(\mathbf{r}, t)$ which appears in these equations is related to the fluctuations of the condensate amplitude and the distribution functions

$$\delta U_{\mathbf{p}}(\mathbf{r}, t) = \frac{1}{2\omega_{\mathbf{p}}} \delta\Pi(\mathbf{r}, t) = \frac{\lambda}{2\omega_{\mathbf{p}}} \left(2\phi_0 \delta\phi(\mathbf{r}, t) + \delta\langle\tilde{\phi}^2(\mathbf{r}, t)\rangle \right) \tag{155}$$

The linearized Klein-Gordon equation (148) and the linearized Vlasov equations (153) and (154), supplemented by the “constitutive relations” (155) and (150), form a closed set of equations to determine a small fluctuation propagating the system in equilibrium.

Noting that $\delta\phi(\mathbf{r}, t)$ and $\delta f(\mathbf{p}, \mathbf{r}, t)$ are real functions, we seek a solution in the form:

$$\delta\phi(\mathbf{r}, t) = \delta\phi e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_+ t)} + \delta\phi^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_- t)}, \tag{156}$$

$$\delta f(\mathbf{p}, \mathbf{r}, t) = \delta f_{\mathbf{p}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_+ t)} + \delta f_{\mathbf{p}}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_- t)}, \tag{157}$$

$$\delta g(\mathbf{p}, \mathbf{r}, t) = \delta g_{\mathbf{p}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_+ t)}, \tag{158}$$

$$\delta\bar{g}(\mathbf{p}, \mathbf{r}, t) = \delta g^*(\mathbf{p}, \mathbf{r}, t) = \delta g_{\mathbf{p}}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_- t)}, \tag{159}$$

where we have introduced the Landau prescription [39],

$$\omega_{\pm} = \omega \pm i\epsilon \quad (160)$$

with a positive infinitesimally small constant ϵ , to set an adiabatic switching-on of the fluctuation, namely $\delta f(\mathbf{p}, \mathbf{r}, t)$, $\delta g(\mathbf{p}, \mathbf{r}, t)$ all vanish slowly as $t \rightarrow -\infty$. The constitutive relations for the fluctuations (150) and the mean field potential (155) now read

$$\begin{aligned} \delta \langle \hat{\phi}^2(\mathbf{r}, t) \rangle = & \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left(\delta f_{\mathbf{p}} + \delta f_{-\mathbf{p}} + \delta g_{\mathbf{p}} + \delta g_{-\mathbf{p}}^* \right) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_+ t)} \\ & + \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left(\delta f_{\mathbf{p}}^* + \delta f_{-\mathbf{p}}^* + \delta g_{\mathbf{p}}^* + \delta g_{-\mathbf{p}} \right) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_- t)} \end{aligned} \quad (161)$$

and

$$\begin{aligned} \delta U_{\mathbf{p}}(\mathbf{r}, t) = & \frac{\lambda}{2\omega_{\mathbf{p}}} \left[2\phi_0 \delta \phi + \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left(\delta f_{\mathbf{p}} + \delta f_{-\mathbf{p}} + \delta g_{\mathbf{p}} + \delta g_{-\mathbf{p}}^* \right) \right] e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_+ t)} \\ & + \frac{\lambda}{2\omega_{\mathbf{p}}} \left[2\phi_0^* \delta \phi^* + \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left(\delta f_{\mathbf{p}}^* + \delta f_{-\mathbf{p}}^* + \delta g_{\mathbf{p}}^* + \delta g_{-\mathbf{p}} \right) \right] e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_- t)} \end{aligned} \quad (162)$$

respectively.

Using these relations, we find from the linearized Klein-Gordon equation (148),

$$\left[-\omega_+^2 + \mathbf{k}^2 + \mu^2 \right] \delta \phi = -\frac{1}{2} \lambda \phi_0 \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left[\delta f_{\mathbf{p}} + \delta f_{-\mathbf{p}} + \delta g_{\mathbf{p}} + \delta g_{-\mathbf{p}}^* \right], \quad (163)$$

and its complex conjugate relation, where we have used the equilibrium relation

$$\mu^2 = m^2 + \frac{\lambda}{2} \left(\phi_0^2 + \langle \tilde{\phi}^2 \rangle_{\text{eq.}} \right) \quad (164)$$

Also, from the linearized Vlasov equations we obtain

$$\begin{aligned} (\omega_+ - \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k}) \delta f_{\mathbf{p}} = & \frac{\lambda}{2\omega_{\mathbf{p}}} (\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k}) \beta \left[2\phi_0 \delta \phi + \sum_{\mathbf{p}'} \frac{1}{2\omega_{\mathbf{p}'}} \left(\delta f_{\mathbf{p}'} + \delta f_{-\mathbf{p}'} + \delta g_{\mathbf{p}'} + \delta g_{-\mathbf{p}'}^* \right) \right] \\ & \times (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p}) \end{aligned} \quad (165)$$

and

$$(-\omega_+ + 2\omega_{\mathbf{p}}) \delta g_{\mathbf{p}} = -\frac{\lambda}{\omega_{\mathbf{p}}} \left[2\phi_0 \delta\phi + \sum_{\mathbf{p}'} \frac{1}{2\omega_{\mathbf{p}'}} \left(\delta f_{\mathbf{p}'} + \delta f_{-\mathbf{p}'} + \delta g_{\mathbf{p}'} + \delta g_{-\mathbf{p}'}^* \right) \right] f_{\text{eq.}}(\mathbf{p}) \quad (166)$$

where we have used

$$\nabla_{\mathbf{p}} \omega_{\mathbf{p}} = \frac{\mathbf{p}}{\omega_{\mathbf{p}}} = \mathbf{v}_{\mathbf{p}} \quad (167)$$

$$\nabla_{\mathbf{p}} f_{\text{eq.}}(\mathbf{p}) = -\mathbf{v}_{\mathbf{p}} \beta (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p}) \quad (168)$$

Using (163), we eliminate $\delta\phi$ in (165) and (166) and find

$$\begin{aligned} (\omega_+ - \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k}) \delta f_{\mathbf{p}} &= \frac{\lambda}{2\omega_{\mathbf{p}}} (\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k}) \beta (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p}) \\ &\quad \times \left(1 + \frac{\lambda \phi_0^2}{\omega_+^2 - \mathbf{k}^2 - \mu^2} \right) \mathcal{F} \end{aligned} \quad (169)$$

$$(\omega_+ - 2\omega_{\mathbf{p}}) \delta g_{\mathbf{p}} = \frac{\lambda}{\omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \left(1 + \frac{\lambda \phi_0^2}{\omega_+^2 - \mathbf{k}^2 - \mu^2} \right) \mathcal{F} \quad (170)$$

where we have introduced the notation

$$\mathcal{F} = \sum_{\mathbf{p}} \frac{1}{2\omega_{\mathbf{p}}} \left(\delta f_{\mathbf{p}} + \delta f_{-\mathbf{p}} + \delta g_{\mathbf{p}} + \delta g_{-\mathbf{p}}^* \right) \quad (171)$$

for the deviation of the fluctuation. Solving (169) and (170) for $\delta f_{\mathbf{p}}$ and $\delta g_{\mathbf{p}}$ respectively, and inserting the results into (171) we obtain the relation

$$\mathcal{F} = \Omega(\omega_+, \mathbf{k}) \mathcal{F} \quad (172)$$

where

$$\begin{aligned} \Omega(\omega_+, \mathbf{k}) &= \frac{\lambda}{2} \left(1 + \frac{\lambda \phi_0^2}{\omega_+^2 - \mathbf{k}^2 - \mu^2} \right) \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{\mathbf{p}}^2} \\ &\quad \times \left[\frac{4\omega_{\mathbf{p}} f_{\text{eq.}}(\mathbf{p})}{\omega_+^2 - 4\omega_{\mathbf{p}}^2} + \frac{\beta (\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k})^2}{\omega_+^2 - (\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k})^2} (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p}) \right]. \end{aligned} \quad (173)$$

Applying the prescription,

$$\lim_{\epsilon \rightarrow +0} \frac{1}{\omega - \omega_0 + i\epsilon} = \mathcal{P} \frac{1}{\omega - \omega_0} - i\pi \delta(\omega - \omega_0) \quad (174)$$

with \mathcal{P} implying that the principal part should be taken in integration, we find that $\Omega(\omega_+, \mathbf{k})$ consists of the real and imaginary parts:

$$\Omega(\omega_+, \mathbf{k}) = \Omega_1(\omega, \mathbf{k}) + i\Omega_2(\omega, \mathbf{k}) \quad (175)$$

The real part is given by

$$\Omega_1(\omega, \mathbf{k}) = -\frac{\lambda}{2} \left(1 + \mathcal{P} \frac{\lambda \phi_0^2}{\omega^2 - \mathbf{k}^2 - \mu^2} \right) \Phi_1(\omega, k) \quad (176)$$

with

$$\Phi_1(\omega, k) = -\mathcal{P} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{\mathbf{p}}^2} \left[\frac{4\omega_{\mathbf{p}} f_{\text{eq.}}(\mathbf{p})}{\omega^2 - 4\omega_{\mathbf{p}}^2} + \frac{\beta(\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k})^2}{\omega^2 - (\mathbf{v}_{\mathbf{p}} \cdot \mathbf{k})^2} (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p}) \right], \quad (177)$$

while the imaginary part is given by

$$\begin{aligned} \Omega_2(\omega, \mathbf{k}) = & -\frac{\lambda}{4} \left(1 + \mathcal{P} \frac{\lambda \phi_0^2}{\omega^2 - \mathbf{k}^2 - \mu^2} \right) \Phi_2(\omega, k) \\ & + \frac{\lambda^2 \phi_0^2}{4\omega_{\mathbf{k}}} \pi (\delta(\omega - \omega_{\mathbf{k}}) - \delta(\omega + \omega_{\mathbf{k}})) \Phi_1(\omega_{\mathbf{k}}, k) \end{aligned} \quad (178)$$

with

$$\begin{aligned} \Phi_2(\omega, k) = & \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{\mathbf{p}}^2} [2\pi (\delta(\omega - 2\omega_{\mathbf{p}}) - \delta(\omega + 2\omega_{\mathbf{p}})) f_{\text{eq.}}(\mathbf{p}) \\ & + \pi \beta \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k} (\delta(\omega - \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k}) - \delta(\omega + \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k})) (1 + f_{\text{eq.}}(\mathbf{p})) f_{\text{eq.}}(\mathbf{p})] \end{aligned} \quad (179)$$

where the integration can be carried out analytically, yielding

$$\Phi_2(\omega, \mathbf{k}) = \frac{1}{8\pi^2} \frac{\omega}{k} \frac{1}{e^{\frac{\mu\beta}{\sqrt{1-(\omega/k)^2}}} - 1} \theta(k - \omega) + \frac{\sqrt{\omega^2 - 4\mu^2}}{2\pi\omega} \frac{1}{e^{\omega\beta/2} - 1} \theta(\omega - 2\mu) \quad (180)$$

for $\omega > 0$. The values of this function for $\omega < 0$ can be found by noting that it is an odd function of ω .

The first term in (180) corresponds to the space-like (scattering) continuum with $\omega = \omega_{\mathbf{p}+\mathbf{k}} - \omega_{\mathbf{p}} \simeq \mathbf{v}_{\mathbf{p}} \cdot \mathbf{k} < k$, while the second term corresponds to the continuum of thermally induced pair creation/annihilation of mesons with energy $\omega > 2\omega_{\mathbf{k}/2} \simeq 2\mu$. The function $\Phi_2(\omega, k)$ (and $\Omega_2(\omega, k)$) has non-vanishing supports in the regions on (ω, k) plane as indicated by the shaded areas in Fig. 2. We plotted in Fig. 3 $\Phi_1(\omega, k)$ and $\Phi_2(\omega, k)$ as functions of ω/k at $k = 0.5\mu$ for two different values of μ/T .

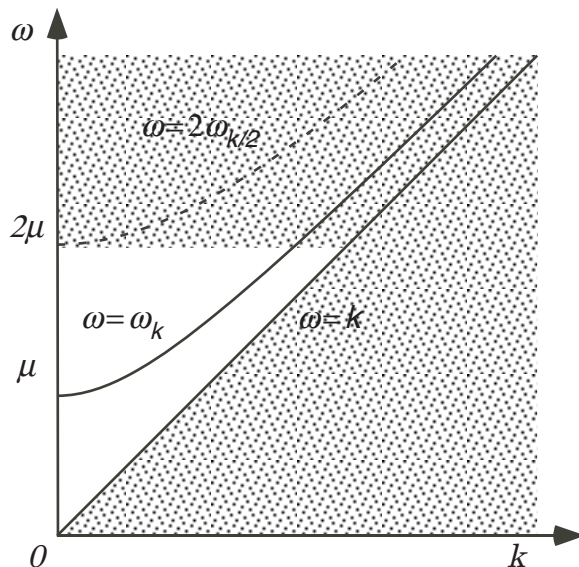


Fig. 2. The regions where $\Omega_2(\omega, k)$ and $\Phi_2(\omega, k)$ have non-vanishing value is shown by shaded areas (two-quasi-particle continua). The solid hyperbola corresponding to the meson poles ($\omega = \omega_{\mathbf{k}}$) on which $\Omega_2(\omega, k)$ has a δ -function singularity in the low temperature phase. The shaded area in the time-like region below the curve $\omega = 2\omega_{\mathbf{k}/2}$ is only an artifact of the long wavelength approximation.

The condition

$$1 = \Omega(\omega, \mathbf{k}) \quad (181)$$

which follows from (172) determines the dispersion relation of a possible long wavelength collective excitation of the system.

Before examining the solutions of the dispersion relation (181), we note that at zero temperature, namely in the absence of quasi-particle excitations, the only solution of our coupled kinetic equations is the one which satisfies the linearized Klein-Gordon equation (148) or (163) with $\delta f = \delta g = 0$ and $\mu = \sqrt{m^2 + \lambda\phi_0^2/2}$. This gives a simple meson pole $\omega = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}$ which appears in the time-like region ($\omega > k$). There is no space-like mode

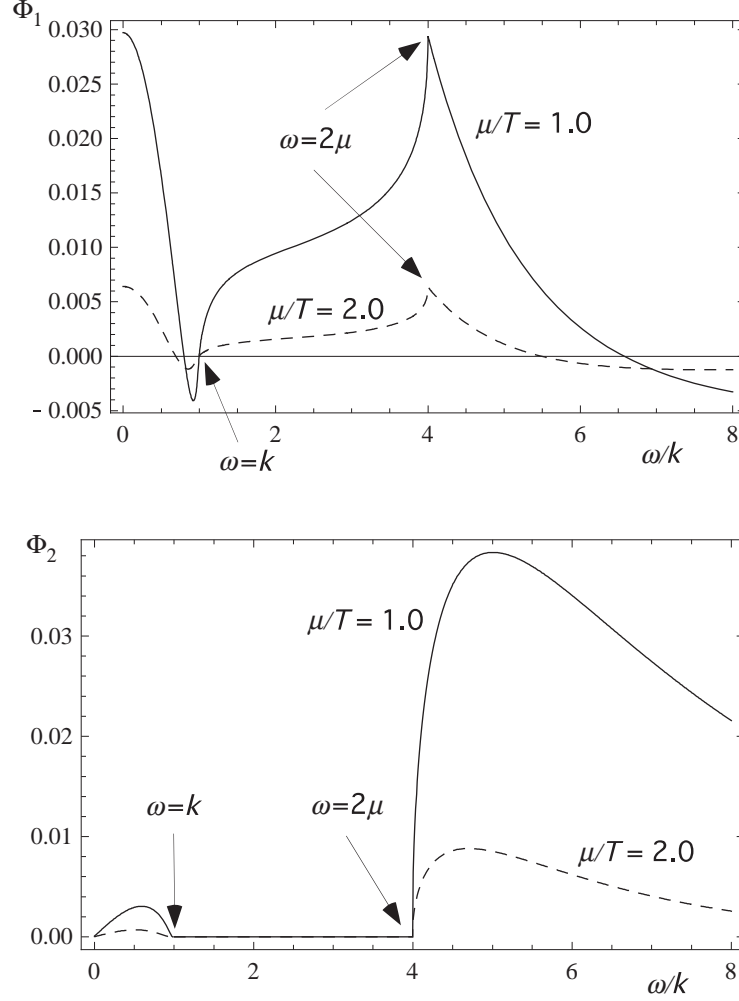


Fig. 3. $\Phi_1(\omega, k)$ and $\Phi_2(\omega, k)$ as a function of ω/k at $k/\mu = 0.5$ and $\mu/T = 1$ (dashed line) , $\mu/T = 2$ (solid line). Both Φ_1 and Φ_2 vanish at $\omega = k$. Φ_2 also vanishes at the pair creation threshold $\omega = 2\mu$ (in the long wavelength approximation) while Φ_1 has a cusp at this point.

of excitations in the absence of quasi-particle excitations due to the mass gap ($\Delta\omega = 2\mu$) for the excitations of the vacuum.

The situation is different in the case of the ordinary Bose-Einstein Condensate (BEC) which possesses, even at zero temperature, a low-lying acoustic mode (Bogoliubov phonon) in the mean-field approximation to repulsive two-body contact interaction[41, 14]. The ordinary BEC contains space-like quasi-particle excitations even at absolute zero temperature which corresponds to the excitation of one of the zero energy particles forming the condensate. The Bogoliubov phonon is a collective excitation of such space-like quasi-particle excitation modes modified by the repulsive mean-field interaction.

At finite non-zero temperature, the system contains continuum of the space-like excitations for all (ω, \mathbf{k}) satisfying $\omega < k$ as signified by the non-vanishing value of $\Omega_2(\omega, \mathbf{k})$ in addition to the continuum in the time-like region $\omega > 2\mu$. (See Fig. 2) Any solution of (181) in the space-like region therefore is subject to the collisionless dissipation known as the Landau damping[39].

We plot the function $\Omega_1(\omega, \mathbf{k})$ in Fig. 4 as a function of ω at fixed value of $k = 0.5\mu$. In the low temperature phase, it contains a singularity at the position of the mesonic pole at $\omega = \sqrt{k^2 + \mu^2}$ while this singularity disappears in the high temperature phase. In the presence of the condensate, a disturbance in the quasi-particle distribution may be absorbed into an excitation of the condensate which then propagates with the mesonic dispersion relation and is converted back again to the quasi-particle excitations. This coupling between the condensate and the quasiparticle excitations generates a collective mode in the low temperature phase.⁵ In the high temperature phase, the quasi-particle excitations couple each other only through their direct interaction, hence no mixing with single mesonic pole.

The high temperature behavior of Ω_1 is the same as that of the function Φ_1 : it is negative at $\omega = 0$ and increases with ω and reaches a positive maximum slightly below $\omega = k$. In the low temperature phase, $\Omega_1(\omega, \mathbf{k})$ reverts the sign for ω below this singularity. One may interpret that effective coupling strength

$$\lambda'(\omega, k) = \lambda \left(1 + \frac{\lambda\phi_0^2}{\omega^2 - \mathbf{k}^2 - \mu^2} \right) = \lambda \left(1 + \frac{3\mu^2}{\omega^2 - \mathbf{k}^2 - \mu^2} \right) \quad (182)$$

changes its sign below the meson pole.⁶ Note that here we have used the relation $\lambda\phi_0^2 = 3\mu^2$ (111) for the condensate amplitude. We note that $\Omega_1(\omega, k)$ always vanishes at $\omega = k$ since $\Phi_1(k, k) = 0$. The cusp at $\omega = 2\mu$ appears at the threshold of the two quasiparticle creation. The condition (181) is fulfilled only near the meson pole in the low temperature phase and it gives a shift of the mesonic excitation spectrum. We could not find any additional solution satisfying (181).

We like to note here that without the coupling to the fluctuations to pair creation or annihilation, we would not have gotten the first term in the function $\Phi_1(\omega, k)$ and then $\Omega_1(\omega, k)$ would have become 1 at $\omega/k > 1$ creating a un-

⁵ This mode may be compared to the collisionless acoustic mode (the quasi-particle sound [48]) which appears in the superfluid ^4He due to the excitation of the condensate coupled with quasi-particle excitations.

⁶ This behavior reminds us of the change of the effective two-body interaction in the Bose-Einstein condensates as a function of the external magnetic field due to the coupling to the intermediate atomic resonance state, the phenomenon known as the Feshbach resonance [14].

damped tachyonic sound mode. Hence the off-diagonal Wigner functions plays an important role in making our framework consistent with causality.

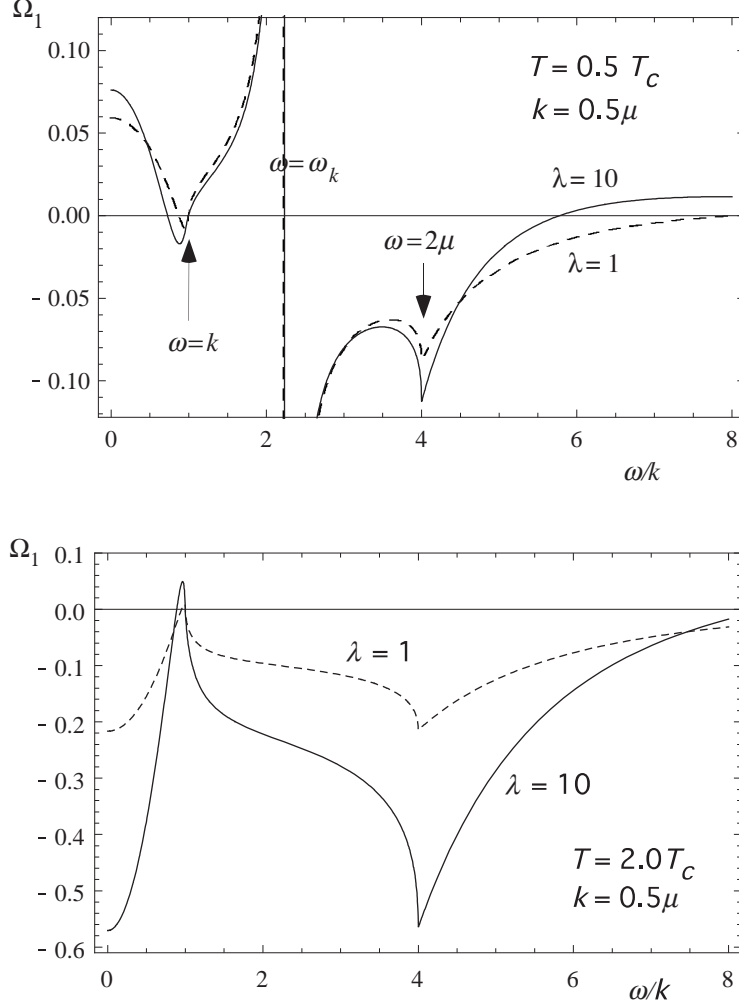


Fig. 4. $\Omega_1(\omega, k)$ as a function of ω/k at fixed k

We introduce the "response function" defined by

$$R(\omega, k) = -\text{Im} \left[\frac{1}{1 - \Omega(\omega, \mu)} \right] = -\frac{\Omega_2(\omega, \mu)}{[1 - \Omega_1(\omega, \mu)]^2 + [\Omega_2(\omega, \mu)]^2} \quad (183)$$

It is plotted in Fig. 5 as a function of ω and compared with the "bare" response function given by

$$R_0(\omega, k) = -\Omega_2(\omega, \mu) \quad (184)$$

The response function in the space-like momentum region gives the dynamic form factor of the cross section of the scattering of particles coupled to the

excitations of the system [46, 47], while its time-like component may give the rate of pair annihilation of the quasi-particles. We note that in the low temperature phase the response function changes its sign below the meson pole due to the sign change of the effective coupling (182). We observe some enhancement (depletion) of the strength in the time-like pair annihilation near threshold and small depletion (enhancement) in the space-like region in the low (high) temperature phase.

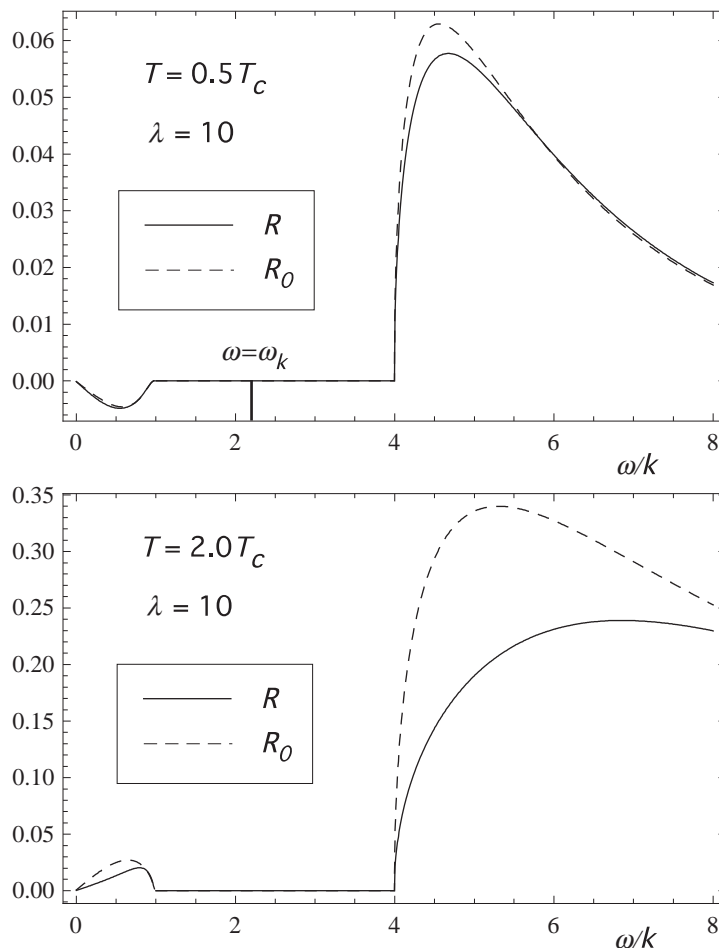


Fig. 5. Plot of the response function $R(\omega, k)$ (184) (solid lines) at $k/\mu = 0.5$. For comparison the bare response function $R_0(\omega, k) = -\Omega_2(\omega, k)$ is also shown (dashed lines). Note that in the low temperature phase R changes sign in the space-like region and at the meson pole ($\omega \simeq \omega_{\mathbf{k}}$) due to the sign change of the effective coupling strength (182).

As we have noted, in the low temperature phase with non-vanishing condensate amplitude ϕ_0 , the mesonic excitations couple with the quasi-particle excitations and this give the additional shift of the meson mass from the one obtained by the gap equation. The shift of the meson pole from the bare spectrum $\omega = \omega_{\mathbf{k}} = \sqrt{k^2 + \mu^2}$ may be computed by the condition

$$\omega^2 - k^2 - \mu^2 - (\omega^2 - k^2 - \mu^2)\Omega(\omega, k) = 0 \quad (185)$$

which in the long wavelength limit ($k = 0$) gives a solution at $\omega^2 = \mu'^2$. The shift of the meson mass $\Delta\mu^2 = \mu'^2 - \mu^2$ is determined by

$$\Delta\mu^2 = \mu'^2 - \mu^2 = 2\lambda \left(\Delta\mu^2 + 3\mu^2 \right) \int \frac{d^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}} \frac{f_{\text{eq.}}(\mathbf{p})}{\mu^2 + \Delta\mu^2 - 4\omega_{\mathbf{p}}^2} \quad (186)$$

where we have used $\lambda\phi_0^2 = 3\mu^2$. In the case of $|\Delta\mu^2| \ll \mu^2$ this may be solved approximately

$$\Delta\mu^2 \simeq 6\lambda\mu^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}} \frac{f_{\text{eq.}}(\mathbf{p})}{\mu^2 - 4\omega_{\mathbf{p}}^2} \bigg/ \left(1 + 2\lambda \int \frac{d^3\mathbf{p}'}{(2\pi)^3 \omega_{\mathbf{p}'}} \frac{f_{\text{eq.}}(\mathbf{p}')}{\mu^2 - 4\omega_{\mathbf{p}'}^2} \right) \quad (187)$$

The equation (186) may be solved by iteration starting from the first approximation (187) by inserting it for $\Delta\mu^2$ in the integrand on the right hand side. We plot in Fig. 6 the meson mass shift determined by this method. The meson mass becomes smaller as T increases and eventually becomes zero at the temperature which satisfies the condition

$$1 = \lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{\mathbf{p}}^3} f_{\text{eq.}}(\mathbf{p}) \quad (188)$$

The vanishing of the effective mass of mesonic excitation may be identified as the onset of instability in the metastable low temperature phase by small fluctuation (spinodal decomposition). The same instability occurs when one approaches to $T = T_c$ from high temperature phase.

We note that the temperature T_{sp} at the spinodal point given by (188) does not coincide with the "backbending" temperature T_1 where $d\mu/dT$ diverges. The latter temperature is determined by the condition

$$1 = \lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2p^2 \omega_{\mathbf{p}}} f_{\text{eq.}}(\mathbf{p}) \quad (189)$$

which can be obtained from the gap equation (110). We found that T_1 is slightly above T_{sp} and the unstable region appears associated with the lower solutions μ of the gap equation. These unstable solutions thus appear only in the region which is not easily accessible and may well be considered as another pathology of the mean field approximation.⁷

⁷ These two temperatures may coincide with T_c in the case of the second order transition of the Landau-type. Such behavior has been obtained in [36] by a temperature-dependent loop expansion method formulated in [35].

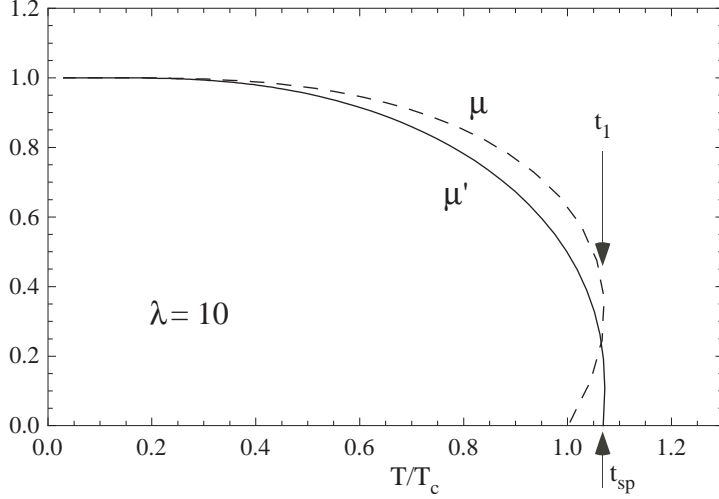


Fig. 6. Mass shift of the mesonic excitation: solid line (μ') is the mass of the excitation determined by (186) and the dashed line is the solution of the gap equation evaluated at $\lambda = 10$. The slopes of two curves diverge at the temperature $t_1 = T_1/T_c = 1.0714$ while μ' vanishes at $t_{sp} = T_{sp}/T_c = 1.0702$.

7 Concluding remarks

In this paper, we have developed a kinetic theory for a system of interacting quantum fields in the mean field approximation taking into account the existence of quasi-particle excitations. We have obtained a set of coupled equations of motion, one for the meson condensate in the form of non-linear Klein-Gordon equation which contains extra non-linearity due to the particle excitations. The equations of motion for quasi-particle excitations are described in terms of the Wigner functions, which reduce to a semiclassical Vlasov equation for one-particle distribution function with a modification due to the coherent pair-creation and pair-annihilation expressed by the off-diagonal components in the Wigner function. These off-diagonal components may be eliminated by a suitable Bogoliubov transformation of the particle creation and annihilation operators for a uniform, time-independent system. However, they remain non-vanishing in general non-uniform, time-dependent systems.

We have shown that in equilibrium these equations are reduced to a gap equation in the Hartree approximation. This implies that our kinetic equations are natural extension of the Hartree approximation to the non-equilibrium situation. It is well known, however, that in this approximation the phase transition becomes first order. This is a generic feature of the mean field approximation [27, 37, 34] which persists also for multi-components scalar fields with continuous $O(N)$ symmetry for arbitrary finite integer N .

In the present work, we studied also the excitations spectrum in the system near equilibrium. In the high temperature phase there is no collective ex-

citation mode in the system besides the two continua of the quasi-particle excitations in the entire space-like energy-momentum region and the time-like region above the pair creation threshold. In the low temperature phase we found that coupling of the meson pole to the quasi-particle continua give rise to the shift of the meson mass which becomes zero at the edge of the spinodal instability line of the first order transition. The spinodal point appears deep inside the meta-stable region of the first order transition which may well be an artifact of the mean field approximation. We found that the coupling to the off-diagonal components of the Wigner functions should be properly included to avoid the appearance of the undamped tachyonic mode.

In this work we used a single component real scalar field model which possesses only discrete symmetry. It is straightforward to extend the present analysis to models with continuous symmetry such as the sigma model with $O(N)$ symmetry. Basic features of the present analysis is preserved in such extension. It is known however that the Goldstone theorem is apparently violated in this approximation for a system with continuous symmetry. We will show in the forthcoming paper [22], that the missing Nambu-Goldstone mode may be retrieved in the collective excitations of the system.

Our coupled kinetic equations may be solved for an arbitrary initial conditions. We plan to study the freezeout dynamics with these equations with more realistic interactions. It would be interesting to see in particular how much flow is generated by the acceleration by the mean field as the vacuum condensate is restored.

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A Proof of the equivalence of the Gaussian density matrix average and the mean field approximation.

Here we present some details of the computation of the equation of motion of the Wigner functions and show that with the Gaussian Ansatz for the density matrix the result is equivalent to what we obtain from the mean-field Hamiltonian (68).

We like to compute the time-derivative of the operator product $a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}$ which appears in the definition of $F(\mathbf{p}, \mathbf{k}, t)$ with $\mathbf{p}_1 = \mathbf{p} + \mathbf{k}/2$ and $\mathbf{p}_2 = \mathbf{p} - \mathbf{k}/2$.

$$\begin{aligned} i \frac{\partial}{\partial t} (a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}) &= i \dot{a}_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} + i a_{\mathbf{p}_1}^\dagger \dot{a}_{\mathbf{p}_2} \\ &= [a_{\mathbf{p}_1}^\dagger, H] a_{\mathbf{p}_2} + a_{\mathbf{p}_1}^\dagger [a_{\mathbf{p}_2}, H] \end{aligned} \quad (\text{A.1})$$

The commutators $[a_{\mathbf{p}_1}^\dagger, H]$ is decomposed into a sum of the commutators with H_i ($i = 1, \dots, 4$) among which the commutators with H_1 and H_3 do not survive the average with the Gaussian density matrix since they only contain the odd power of the field operators. We only need to compute the commutators with H_2 and H_4 .

To compute the commutator with H_2 it is convenient to rewrite H_2 as

$$H_2 = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \right) + \frac{\lambda}{4} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \tilde{\phi}^2(\mathbf{r}, t) \quad (\text{A.2})$$

We then find

$$\begin{aligned} [a_{\mathbf{p}}(t), H_2] &= \omega_{\mathbf{p}} a_{\mathbf{p}}(t) + \frac{\lambda}{4} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) [a_{\mathbf{p}}(t), \tilde{\phi}^2(\mathbf{r}, t)] \\ &= \omega_{\mathbf{p}} a_{\mathbf{p}}(t) + \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \frac{e^{-i\mathbf{p} \cdot \mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} \tilde{\phi}(\mathbf{r}, t) \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} [a_{\mathbf{p}}^\dagger(t), H_2] &= -\omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger(t) + \frac{\lambda}{4} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) [a_{\mathbf{p}}^\dagger(t), \tilde{\phi}^2(\mathbf{r}, t)] \\ &= -\omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger(t) - \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} \tilde{\phi}(\mathbf{r}, t) \end{aligned} \quad (\text{A.4})$$

where we have used the following formulae:

$$[a_{\mathbf{p}}^\dagger, \tilde{\phi}^n(\mathbf{r}, t)] = n \tilde{\phi}^{n-1}(\mathbf{r}, t) [a_{\mathbf{p}}^\dagger, \tilde{\phi}(\mathbf{r}, t)] = -n \tilde{\phi}^{n-1}(\mathbf{r}, t) \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} \quad (\text{A.5})$$

$$[a_{\mathbf{p}}, \tilde{\phi}^n(\mathbf{r}, t)] = n\phi^{n-1}(\mathbf{r}, t)[a_{\mathbf{p}}, \tilde{\phi}(\mathbf{r}, t)] = n\phi^{n-1}(\mathbf{r}, t) \frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} \quad (\text{A.6})$$

With these results we obtain

$$\begin{aligned} [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_2] &= -(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} \\ &\quad - \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \tilde{\phi}(\mathbf{r}, t) \frac{e^{i\mathbf{p}_1\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_1}}} a_{\mathbf{p}_2} \\ &\quad + \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \tilde{\phi}(\mathbf{r}, t) \frac{e^{-i\mathbf{p}_2\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_2}}} a_{\mathbf{p}_1}^\dagger \end{aligned} \quad (\text{A.7})$$

The Gaussian average of this equation gives

$$\begin{aligned} \langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_2] \rangle &= -(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \langle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} \rangle \\ &\quad - \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \frac{e^{i\mathbf{p}_1\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_1}}} \langle \tilde{\phi}(\mathbf{r}, t) a_{\mathbf{p}_2} \rangle \\ &\quad + \frac{\lambda}{2} \int d\mathbf{r} \phi_c^2(\mathbf{r}, t) \frac{e^{-i\mathbf{p}_2\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_2}}} \langle \tilde{\phi}(\mathbf{r}, t) a_{\mathbf{p}_1}^\dagger \rangle \end{aligned} \quad (\text{A.8})$$

Next we compute the commutators with H_4 . We first compute

$$[a_{\mathbf{p}}(t), H_4] = \frac{\lambda}{3!} \int d\mathbf{r} \tilde{\phi}^3(\mathbf{r}, t) \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}}(t) \quad (\text{A.9})$$

$$[a_{\mathbf{p}}^\dagger(t), H_4] = -\frac{\lambda}{3!} \int d\mathbf{r} \tilde{\phi}^3(\mathbf{r}, t) \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}}(t) \quad (\text{A.10})$$

We then use these results to obtain

$$\begin{aligned} [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_4] &= -\frac{\lambda}{3!} \int d\mathbf{r} \tilde{\phi}^3(\mathbf{r}, t) \frac{e^{i\mathbf{p}_1\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_1}}} a_{\mathbf{p}_2} \\ &\quad + \frac{\lambda}{3!} \int d\mathbf{r} \tilde{\phi}^3(\mathbf{r}, t) \frac{e^{-i\mathbf{p}_2\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_2}}} a_{\mathbf{p}_1}^\dagger \end{aligned} \quad (\text{A.11})$$

Taking the average with the Gaussian density matrix, we find

$$\begin{aligned} \langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_4] \rangle &= -\frac{\lambda}{2} \int d\mathbf{r} \langle \tilde{\phi}^2(\mathbf{r}, t) \rangle \frac{e^{i\mathbf{p}_1\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_1}}} \langle \tilde{\phi}(\mathbf{r}, t) a_{\mathbf{p}_2} \rangle \\ &\quad + \frac{\lambda}{2} \int d\mathbf{r} \langle \tilde{\phi}^2(\mathbf{r}, t) \rangle \frac{e^{-i\mathbf{p}_2\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{p}_2}}} \langle \tilde{\phi}(\mathbf{r}, t) a_{\mathbf{p}_1}^\dagger \rangle \end{aligned} \quad (\text{A.12})$$

We observe the similarity between the last two terms of the commutator with H_2 and the commutator with H_4 after Gaussian average. These terms can be combined by introducing the self-energy function,

$$\Pi(\mathbf{r}, t) = \phi_c^2(\mathbf{r}, t) + \langle \tilde{\phi}^2(\mathbf{r}, t) \rangle = \sum_{\mathbf{q}} \Pi_{\mathbf{q}}(t) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (\text{A.13})$$

Integration over the space coordinate \mathbf{r} give a delta function $\delta(\mathbf{q} + \mathbf{p} + \mathbf{p}_1)$ and $\delta(\mathbf{q} + \mathbf{p} - \mathbf{p}_2)$. Performing the integral over \mathbf{p} yields

$$\begin{aligned} \langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H] \rangle &= -(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \langle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} \rangle \\ &\quad - \frac{\lambda}{4} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1+\mathbf{q}} \omega_{\mathbf{p}_2}}} (\langle a_{-\mathbf{p}_1-\mathbf{q}} a_{\mathbf{p}_2} \rangle + \langle a_{\mathbf{p}_1+\mathbf{q}}^\dagger a_{\mathbf{p}_2} \rangle) \\ &\quad + \frac{\lambda}{4} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2-\mathbf{q}}}} (\langle a_{\mathbf{p}_2-\mathbf{q}} a_{\mathbf{p}_1}^\dagger \rangle + \langle a_{-\mathbf{p}_2+\mathbf{q}}^\dagger a_{\mathbf{p}_1}^\dagger \rangle) \end{aligned} \quad (\text{A.14})$$

This result coincides with the average of the commutator with the mean field Hamiltonian H_{mf} defined by (68).

$$\langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H] \rangle = \langle [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_{\text{mf}}] \rangle \quad (\text{A.15})$$

Commutators of the four bilinear operator products with the mean-field Hamiltonian are listed below:

$$\begin{aligned} [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}, H_{\text{mf}}] &= -(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} \\ &\quad - \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1+\mathbf{q}} \omega_{\mathbf{p}_2}}} (a_{-\mathbf{p}_1-\mathbf{q}} a_{\mathbf{p}_2} + a_{\mathbf{p}_1+\mathbf{q}}^\dagger a_{\mathbf{p}_2}) \\ &\quad + \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2-\mathbf{q}}}} (a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2-\mathbf{q}} + a_{\mathbf{p}_1}^\dagger a_{-\mathbf{p}_2+\mathbf{q}}^\dagger) \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} [a_{\mathbf{p}_1} a_{\mathbf{p}_2}, H_{\text{mf}}] &= (\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1} a_{\mathbf{p}_2} \\ &\quad + \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1+\mathbf{q}} \omega_{\mathbf{p}_2}}} (a_{\mathbf{p}_1-\mathbf{q}} + a_{-\mathbf{p}_1+\mathbf{q}}^\dagger) a_{\mathbf{p}_2} \\ &\quad + \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2-\mathbf{q}}}} a_{\mathbf{p}_1} (a_{\mathbf{p}_2-\mathbf{q}} + a_{-\mathbf{p}_2+\mathbf{q}}^\dagger) \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} [a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger, H_{\text{mf}}] &= -(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger \\ &\quad - \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1+\mathbf{q}} \omega_{\mathbf{p}_2}}} (a_{-\mathbf{p}_1-\mathbf{q}} a_{\mathbf{p}_2} + a_{\mathbf{p}_1+\mathbf{q}}^\dagger a_{\mathbf{p}_2}) \end{aligned}$$

$$-\frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2 - \mathbf{q}}}} (a_{\mathbf{p}_2 - \mathbf{q}} a_{\mathbf{p}_1}^{\dagger} + a_{-\mathbf{p}_2 + \mathbf{q}}^{\dagger} a_{\mathbf{p}_1}^{\dagger}) \quad (\text{A.18})$$

$$\begin{aligned} [a_{\mathbf{p}_1} a_{\mathbf{p}_2}^{\dagger}, H_{\text{mf}}] &= (\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) a_{\mathbf{p}_1} a_{\mathbf{p}_2}^{\dagger} \\ &+ \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1 + \mathbf{q}} \omega_{\mathbf{p}_2}}} (a_{\mathbf{p}_1 - \mathbf{q}} + a_{-\mathbf{p}_1 + \mathbf{q}}^{\dagger}) a_{\mathbf{p}_2}^{\dagger} \\ &- \frac{1}{2} \sum_{\mathbf{q}} \frac{\Pi_{\mathbf{q}}}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2 - \mathbf{q}}}} a_{\mathbf{p}_1}^{\dagger} (a_{\mathbf{p}_2 - \mathbf{q}} + a_{-\mathbf{p}_2 + \mathbf{q}}^{\dagger}) \end{aligned} \quad (\text{A.19})$$

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